3.2 Inverse of a Matrix

Suggested exercises: 1-45 odd, 49

We already saw that matrix multiplication is not commutative: $AB \neq BA$ in general. Next we will discuss another difference between working with matrices and numbers: finding multiplicative inverses. While finding an inverse for a nonzero number is as simple as taking the reciprocal, we must work a bit harder to find the inverse of a matrix. Fortunately, for 2×2 matrices, there is an easy formula.

4 keys:

- Cancellation
- Inverse matrices
- Find the inverse of a matrix
- Inverses and linear systems

I Cancellation

With equations, we can cancel a common factor on both sides of the equation and the equation remains valid. This will not work with matrices.

Example. Let A, B, and C be the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

$$B = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

II Inverse of a matrix

The matrix A in the previous section could not be canceled from both sides because it would make a false statement. However there are many matrices for which cancellation does work, provided we do it the right way. This will involve "dividing" by a matrix, or multiplying by the inverse of a matrix.

Definition The **identity matrix** is the square matrix with 1 on the main diagonal and 0 everywhere else. There is an identity matrix for each dimension. We will write I for the identity matrix, or I_n if we need to note the dimension. For any matrix A, multiplying by I leaves A unchanged.

Example.

Definition A square matrix A is said to be **invertible** if there exists a matrix A^{-1} such that $A^{-1}A = I = AA^{-1}$. The matrix A^{-1} is called the **inverse** of A.

Example.

Property 1 (Properties of inverses) Let A and B be square matrices. Then

(i) If A is invertible, then the inverse A^{-1} is unique.

(ii) If A is invertible, then A^{-1} is invertible, and the inverse of A^{-1} is A.

(iii) If A and B are invertible, then the product AB is invertible with inverse $(AB)^{-1} = B^{-1}A^{-1}$.

Property 2 (Test for inverses) Let A and B be $n \times n$ matrices. If $AB = I$, then $BA = I$ and $B = A^{-1}$.

Example.

Example.

III Finding the inverse of a matrix

Some matrices have inverses, but some do not. How do we know when an inverse exists? And how do we find it? Both questions can be answered in the same way, by looking back at a procedure from Chapter 2: Gauss-Jordan elimination.

Example.

Finding A^{-1}:

1. Form the augmented matrix $[A|I]$ and use row operations to put the matrix in reduced echelon form.
2. If A becomes the identity matrix I in reduced echelon form, then the augmented matrix will be $[I|A^{-1}]$ and A is invertible.
3. If A is not the identity matrix when it is reduced, but rather has one or more rows of zeros at the bottom, then A is not invertible.

Example.

Example.

IV Inverses and linear systems

Instead of using an augmented matrix, we can use matrix multiplication to express a system of linear equations.

Example.

Definition A $n \times 1$ matrix is called a **vector** or a **column vector**.

If the coefficient matrix of our system of equations is invertible, we can use its inverse to solve our system of equations, just like when we solve single equations.

Example.

When we work with 2×2 matrices, there is a nice formula for finding the inverse without having to use augmented matrices. If our matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then we define $D = ad - bc$. Then the matrix A^{-1} can be given by

$$A^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example.

Property 3 If A is an $n \times n$ matrix, then the following statements are equivalent:

(i) The row rank of A is n.

(ii) A is invertible.

(iii) For any vector B, the system $AX = B$ has a unique solution.