3.3 Leontief Open Model

Suggested exercises: 1-17 odd
We now turn to an interesting use of matrices in an economy. The economist who discovered this method was awarded the 1973 Nobel Prize in Economics.

Consumption, Demand, and Production

Previously, we discussed production problems using linear programs, where we needed to find out how to meet the demand for products. In those examples, we focused on one company with several different products. Now we will consider a situation where instead we have several different companies producing, and they depend on each other for resources. The goal is to determine how much of each product should be manufactured so that the consumer demand is met. An example will help illustrate this idea.

Example. Suppose a simple economy has 2 industries: electricity and oil. Producing \$1 of electricy requires \$0.90 of electricity and \$0.20 of oil, while production of \$1 of oil requires \$0.40 of electricity and \$0.10 of oil. The demand for electricity and oil is for \$300 in electricity and \$100 in oil. How much electricity and oil should be produced to meet the demand?

Definition The **consumption matrix** C of an economy organizes the costs for operating a set of industries. The columns represent the costs for each product, and the rows represent the resources necessary for each product. (Think: columns = output, rows = input.) The entries of C are called the **input coefficients**.

Definition The **demand vector** of an economy represents the external demand for the products in the economy, that is, the demand not needed to produce the products themselves.

Definition The **production vector** for an economy holds the variables for the how much of each product that should be made to fulfill the demand. When a solution is found, the vector represents the **production schedule** for the economy.

There are 4 steps involved with finding the production schedule:

1. Find the consumption matrix C.
2. Find $I - C$.
3. Invert $I - C$.
4. The production vector is given by $X = (I - C)^{-1}D$ for a demand vector D.

Example. Suppose that we have an economy with labor, transportation, and food industries. Let \$1 in labor require \$0.40 in transportation and \$0.20 in food, whereas \$1 in transportation takes \$0.50 in labor and \$0.30 in transportation, and \$1 in food production uses \$0.50 in labor, \$0.05 in transportation, and \$0.35 in food. Let the demand for the current production period be \$10,000 labor, \$20,000 transportation, and \$10,000 food. Find the production schedule for the economy.

We say that a matrix is **nonnegative** if all of its entries are nonnegative, and we write $A \geq 0$. For our production vector X, we must have $X \geq 0$ since we are talking about something we are producing. To ensure that we can find a nonnegative production vector, we need to look at our consumption matrix.

Definition We say that the economy and the consumption matrix C are **productive** if $(I - C)^{-1} \geq 0$. In this case, we can find a nonnegative production vector for any demand vector.

So when is our consumption matrix C a productive matrix? The following theorem gives us an easy way to test for a productive matrix.

Property 1 If a consumption matrix C for an economy is such that either (a) the sum of each column is less than 1 or (b) the sum of each row is less than 1, then the economy is productive.

Note that an economy can still be productive even without the (a) or (b), but these conditions guarantee us a productive system.

Example.

Example. A construction company has the products labor and lumber. Let each \$1 in labor require \$0.30 in lumber, while each \$1 in lumber requires \$2 in labor and \$0.10 in lumber. Suppose that there is an outside demand for \$600 labor and \$1200 lumber. How much of each product will be used interally?

Example. A Carribean economy has 2 industries: bananas and coconut oil. Let each \$1 in bananas require \$0.40 in bananas and \$0.10 in oil, while \$1 in coconut oil requires \$0.20 in bananas and \$0.80 in oil. The outside demand is for \$400 in bananas and \$100 in oil. How much of each needs to be produced?

Example. An electrician (E) and carpenter (C) each have a company. For each \$1 of business that E does, he uses \$0.10 of his own services and \$0.20 of C's services. For each \$1 of business that C does, he uses \$0.30 of E's services and \$0.20 of his own. In a certain week, E does \$600 of outside business and C does \$400. Find the consumption matrix C and compute $I - C$, $(I - C)^{-1}$, and the production vector for the given demand.

Example. Show that the following consumption matrix is productive:

$$\begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.1 & 0.1 & 0.7 \\ 0.1 & 0.2 & 0.4 \end{bmatrix}$$