PERMUTATIONS AND COMBINATIONS

Solving problems involving permutations and combinations can be difficult at first, but after some practice, you will notice some patterns of thinking about these problems. Here are some basic guidelines that will help you as you learn to solve these problems.

(1) **Look for independent stages.**
If a problem can be broken into smaller, easier stages, it will be easier to deal with those and combine the answers in the end. Look for choices that are completely independent of each other. Choices that are similar to each other (e.g., choosing where to put the A's from the word $ALABAMA$) can often be grouped into a single stage.

(2) **Decide whether to use permutations or combinations for each stage.**
When you make your choices, does order matter? Will choosing the items in a different order change the end result? Do this for each stage; sometimes a problem will use combinations for one stage, and permutations for another stage.

(3) **Decide whether your ordered samples are with or without replacement.**
Knowing whether there is replacement will affect which formula you use to compute the answer. Sometimes a problem won't state whether there is replacement. Use your judgment to decide whether a choice can be made more than once.

(4) **Use the appropriate formula to compute the number of choices for each stage.**
We have three main formulas, one for ordered samples with replacement, one for $P(n, k)$, and one for $C(n, k)$. Knowing these formulas is the key to getting a numerical answer.

(5) **Use the multiplication principle to find the final answer.**

Somtimes you will find a problem where there is more than one way to list the stages to get to the final answer. In this case, compute the number of ways for each list of stages, and then add your answers together.

Example (from `http://www.ilovemaths.com/3permcomb.htm`*).* Consider all of the numbers from 100 to 999. How many of these numbers are there with

(a) no restrictions?
(b) every digit either a 2 or a 5?
(c) the digit in the hundreds place is 5?
(d) exactly one of the digits is a 5?
(e) at least one of the digits is a 5?
(f) no digit is repeated?
(g) at least one digit is repeated?
(h) the digit in the units place is 5?

Solution.

(a) We can think of this as three stages: pick the hundreds place, pick the tens place, and pick the units place. We can choose any number but 0 for the hundreds place (note that 081 is *not* a number between 100 and 999), so there are 9 choices. All 10 digits are possible for the tens and units places, so we have a total of $9 \cdot 10 \cdot 10 = 900$ numbers.

(b) Since our choices are limited to 2 and 5, we only have two choices for each spot. We can use a digit more than once, so this is an ordered sample with replacement. Thus we have $2^3 = 8$ numbers.

(c) Our hundreds digit is chosen, so we have ten choices each for the remaining two spots. Thus there are $10^2 = 100$ numbers.

(d) There are three types of number with exactly one 5:
- $\underline{5}$ _ _
- _ $\underline{5}$ _
- _ _ $\underline{5}$

In the first case, there are 9 choices for each of the blanks (we can't choose another 5): $9^2 = 81$. In the second and third cases, there are 8 choices for the first blank (we can't choose 0 or 5), and 9 choices for the second: $8 \cdot 9 = 72$. Thus there are $81 + 72 + 72 = 225$ numbers with exactly one 5.

(e) There are two ways to do this problem. We'll start with the "harder" way.

If at least one digit is a 5, we have three ways to go through the stages: exactly one digit is a 5, exactly two digits are 5, and all three digits are 5. For the first, we saw that there are 225 numbers with exactly one 5. We could do the same thing with exactly two 5s:
- $\underline{5}$ $\underline{5}$ _
- $\underline{5}$ _ $\underline{5}$
- _ $\underline{5}$ $\underline{5}$

We find 9 numbers in the first and second cases, and 8 numbers in the third, for a total of 26 numbers. Finally there is one number with exactly three 5s: 555. Adding all of our answers, we see that $225 + 26 + 1 = 252$ numbers have at least one 5.

The easier way to do this is to count the opposite set and subtract it from the answer we got in (a): how many numbers have no 5s? We have 8 choices for the first digit, and 9 choices for each of the second digits: $8 \cdot 9 \cdot 9 = 648$, and $900 - 648 = 252$.

(f) If no digit is repeated, we do not have replacement. So our first digit can be chosen in 9 ways (remember we can't use 0), our second digit in 9 ways (now we can use 0), and our third digit in 8 ways: $9 \cdot 9 \cdot 8 = 648$.

(g) We could break this down into cases just like we did with the 5, but it will be easier to do the subtraction method: there are 900 numbers, and 648 of them don't have any repeated digits. Thus $900 - 648 = 252$ of them do have repeated digits.

(h) Since the units place is chosen for us, we need to choose the hundreds place and the tens place. We have 9 choices for the first, and 10 choices for the second: $9 \cdot 10 = 90$ numbers.