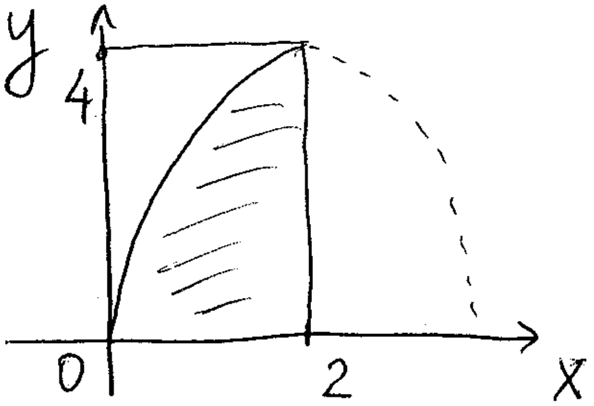
Math 241 **Midterm Exam 3** **Fall 2007**

Name (print in **caps**): SOLUTION Discussion Section: ______

Show all your work. No credit will be given for unjustified answers. Check that your exam contains all the 5 pages. Calculators are not allowed. Good luck!

1. (10 points) Sketch the region of integration and then reverse the order of integration:

$$I = \int_0^2 \int_0^{4x-x^2} f(x,y)\,dy\,dx.$$



$y = 4x - x^2$ parabola

vertex:

$(4x-x^2)' = 4 - 2x = 0$

$x = 2 \quad y = 4$

Solve $y = 4x - x^2$ for x:

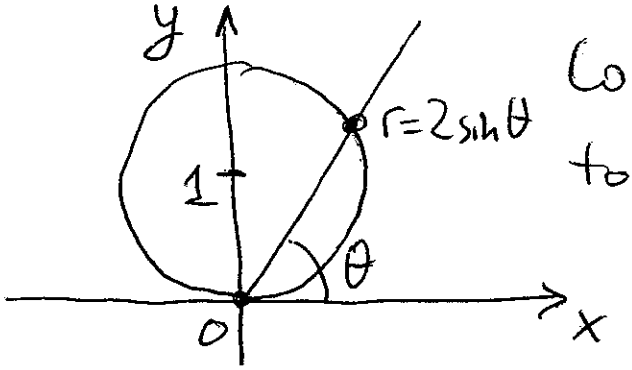
$x^2 - 4x + y = 0$

$x = 2 \pm \sqrt{4-y}$

Left branch: $x = 2 - \sqrt{4-y}$

$$I = \int_0^4 \int_{2-\sqrt{4-y}}^{2} f(x,y)\,dx\,dy$$

2. (10 points) Find the mass of the lamina bounded by the circle $x^2 + (y-1)^2 = 1$ with density $\delta(x, y) = (x^2 + y^2)^{-1/2}$. (Use polar coordinates.)



Convert $x^2+(y-1)^2=1$ to polar coordinates

$$x^2+y^2-2y+\cancel{1}=\cancel{1}$$

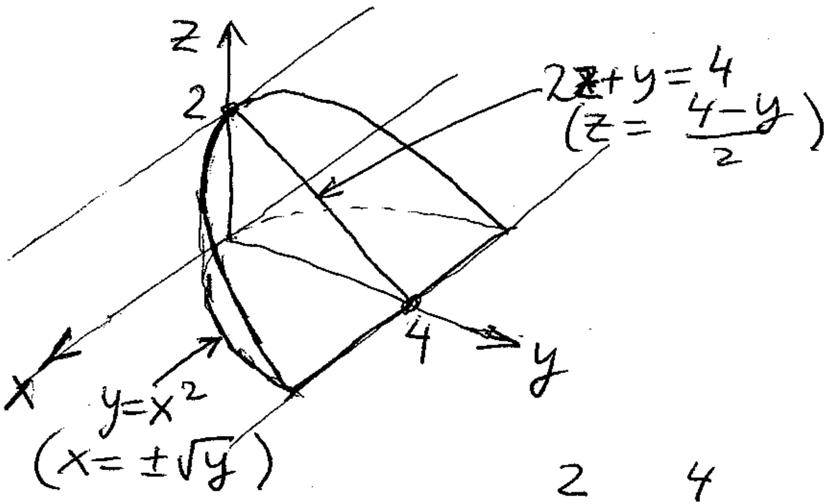
$$r^2-2r\sin\theta=0 \qquad \underline{r=2\sin\theta}.$$

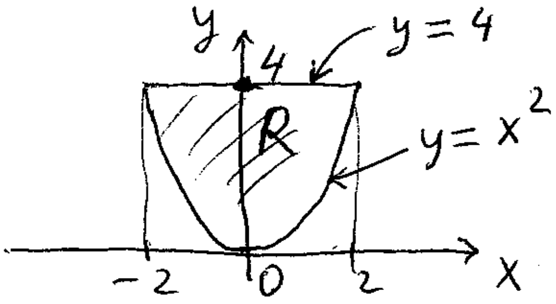
$$m=\iint_R \delta\, dA = \int_0^{\pi}\int_0^{2\sin\theta} r^{-1}\, r\, dr\, d\theta$$

$$= \int_0^{\pi} 2\sin\theta\, d\theta = 4$$

3. (10 points) The solid T is bounded by the surfaces $z = 0$, $y = x^2$, and $2z + y = 4$. Sketch the region T.

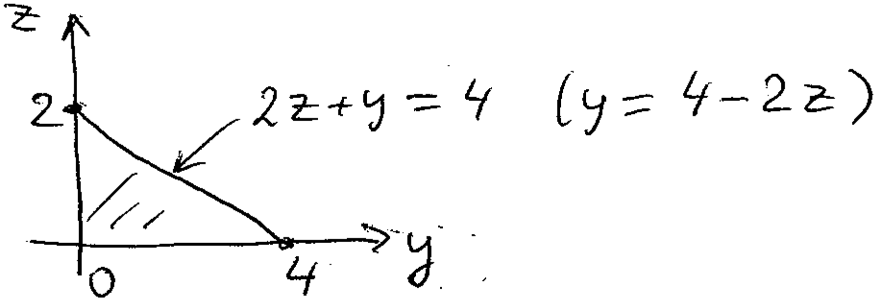
(a) (5 pts) Sketch the projection of T to the xy-plane and express $\int\int\int_T f(x,y,z)\,dV$ as an iterated integral in the order $dz\,dy\,dx$.





$$\iiint_T f\,dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{\frac{4-y}{2}} f(x,y,z)\,dz\,dy\,dx$$

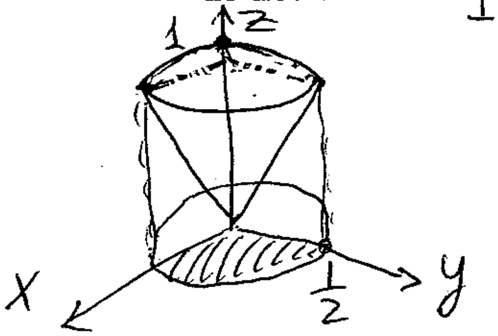
(b) (5 pts) Sketch the projection of T to the yz-plane and express $\int\int\int_T f(x,y,z)\,dV$ as an iterated integral in the order $dx\,dy\,dz$.



$$\iiint_T f\,dV = \int_{0}^{2}\int_{0}^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$$

4. (10 points) Let T be the solid in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$) above the cone $z = \sqrt{3x^2 + 3y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 1$.

(a) (5 pts) Convert $\int\int\int_T \sqrt{x^2+y^2+z^2}\, dV$ to the cylindrical coordinates, but **do not evaluate.** $I=$



Project to xy plane:

$$x^2+y^2+\left(\sqrt{3x^2+3y^2}\right)^2 = 1$$

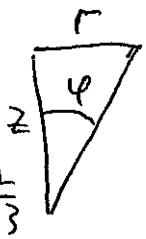
$$4(x^2+y^2) = 1 \qquad r = \frac{1}{2}$$

$$x^2+y^2+z^2 = 1 \iff r^2+z^2 = 1$$

$$z = \sqrt{3x^2+3y^2} \iff z = \sqrt{3}\, r$$

$$I = \int_0^{\pi/2} \int_0^{1/2} \int_{\sqrt{3}\, r}^{\sqrt{1-r^2}} \sqrt{r^2+z^2}\; r\, dz\, dr\, d\theta$$

(b) (5 pts) Convert $\int\int\int_T \sqrt{x^2+y^2+z^2}\, dV$ to the spherical coordinates and evaluate.



$$x^2+y^2+z^2 = 1 \iff \rho = 1$$

$$z = \sqrt{3x^2+3y^2} \iff z = \sqrt{3}\, r \iff \tan\varphi = \frac{1}{\sqrt{3}}$$

$$dV = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \qquad \underline{\varphi = \frac{\pi}{6}}$$

$$I = \int_0^{\pi/2} \int_0^{\pi/6} \int_0^1 \rho\; \rho^2 \sin\varphi\; d\rho\, d\varphi\, d\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\pi/6} \sin\varphi\, d\varphi \int_0^1 \rho^3\, d\rho$$

$$= \frac{\pi}{2} \left[-\cos\varphi\right]_0^{\pi/6} \frac{1}{4} = \frac{\pi}{8}\left(1 - \frac{\sqrt{3}}{2}\right)$$

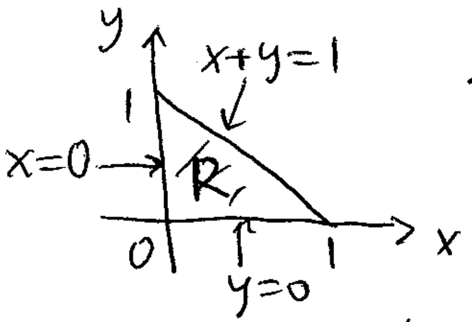
5. (10 points) Let R be the region bounded by the coordinate axes and the line $x+y=1$. Evaluate

$$I = \iint_R e^{(x+y)^2}\, dA$$

by the substitution $u = x+y$, $v = x-y$.

$$x = \frac{u+v}{2} \qquad J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{4} - \frac{1}{4}$$

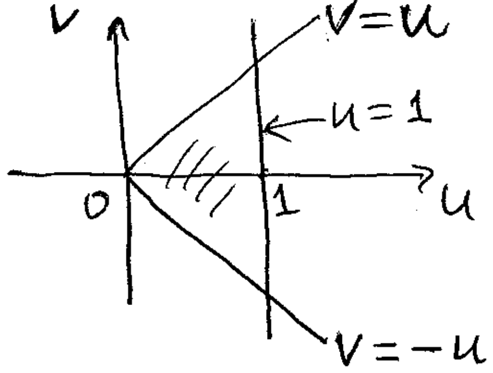
$$y = \frac{u-v}{2} \qquad\qquad = -\frac{1}{2}$$



$x = 0 \iff u+v=0 \quad (v=-u)$

$y = 0 \iff u-v=0 \quad (v=u)$

$x+y=1 \iff u=1$



$$I = \int_0^1 \int_{-u}^{u} e^{u^2}\, \overset{|J|}{\overset{\parallel}{\frac{1}{2}}}\, dv\, du$$

$$= \int_0^1 2u\, e^{u^2} \frac{1}{2}\, du$$

$$= \frac{1}{2} e^{u^2}\Big|_0^1 = \underline{\frac{1}{2}(e-1)}$$

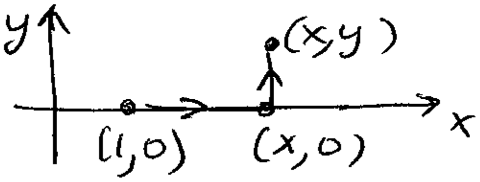
6. (10 points)

(a) Determine whether the vector field $\mathbf{F}(x,y) = \langle \overset{P}{\overset{\|}{x - \frac{y}{x^2}}}, \overset{Q}{\overset{\|}{\frac{1}{x} + 3y^2 + 1}} \rangle$ is conservative in the region $D = \{(x,y) : x > 0\}$ and if so, then find a potential function.

$$P_y = -\frac{1}{x^2} \qquad Q_x = -\frac{1}{x^2} \qquad P_y = Q_x$$

$\mathcal{D}$ is simply connected

Hence $\vec{F}$ is conservative, $\vec{F} = \nabla f$



$$f(x,y) = \int_{(1,0)}^{(x,y)} P\,dx + Q\,dy$$

$$= \int_1^x P(x,0)\,dx + \int_0^y Q(x,y)\,dy$$

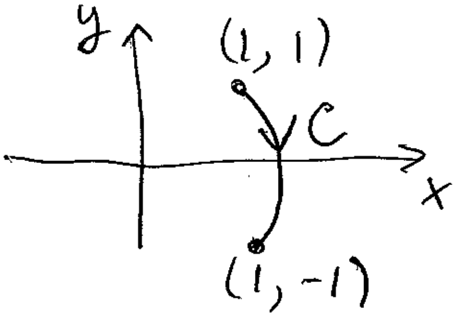
$$= \int_1^x x\,dx + \int_0^y \left(\frac{1}{x} + 3y^2 + 1\right) dy$$

$$= \frac{x^2}{2} - \frac{1}{2} + \frac{y}{x} + y^3 + y$$

Can omit the constant "$-\frac{1}{2}$".

$$f(x,y) = \frac{x^2}{2} + \frac{y}{x} + y^3 + y$$

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C runs from $(1,1)$ to $(1,-1)$ along the arc of the circle $x^2 + y^2 = 2$ in the right half-plane.



$$\int_C \vec{F} \cdot d\vec{r} = f(1,-1) - f(1,1)$$

$$= \left(\frac{1}{2} - 1 - 1 - 1\right) - \left(\frac{1}{2} + 1 + 1 + 1\right)$$

$$= -6$$