Math 221 Exam #1 Solutions

1. Let $h(x) = \begin{cases} 2x+1 & x > 0, \\ 2 - x^2 & x \le 0. \end{cases}$

(a) Does $\lim_{x \to 0^+} h(x)$ exist? If so, what is it?

Using rules for limits of polynomials in the book, $\lim_{x\to 0^+} h(x) = \lim_{x \to 0^+}(2x+1) = 2\cdot 0 + 1 = 1$. *This limit exists.*

(b) Does $\lim_{x \to 0^-} h(x)$ exist? If so, what is it?

Using rules for limits of polynomials in the book, $\lim_{x\to 0^-} h(x) = \lim_{x \to 0^-}(2 - x^2) = 2 - 0^2 = 2$. *This limit exists.*

(c) Does $\lim_{x \to 0} h(x)$ exist? If so, what is it?

Since $\lim_{x\to 0^+} h(x) \neq \lim_{x \to 0^-} h(x)$*, this limit does not exist.*

2. (a) Complete the following sentence with a careful and complete definition:

A sequence (a_n) *of real numbers converges to a real number* L *if for all* $\epsilon > 0$*, there exists a natural number* N *so that* $|a_n - L| < \epsilon$ *for every* $n \ge N$.

(b) Evaluate the following limit: $\lim_{n\to\infty}\left(2 + \frac{3}{2^n+1}\right)$. Justify your answer.

ANSWER #1: *For every* n*, we have* $0 < \frac{3}{2^n+1} < \frac{3}{2^n} = 3(\frac{1}{2})^n$. *As shown in class,* $\lim_{n\to\infty}(\frac{1}{2})^n = 0$ *(since* $\frac{1}{2} < 1$*). By the Squeeze Theorem, since* $\lim_{n\to\infty} 3(\frac{1}{2})^n = 0$*, we have* $\lim_{n\to\infty}\frac{3}{2^n+1} = 0$. *By the limit laws,*

$$\lim_{n\to\infty}\left(2 + \frac{3}{2^n+1}\right) = 2 + \lim_{n\to\infty}\frac{3}{2^n+1} = 2 + 0 = 2.$$

ANSWER #2: *We claim that the limit is equal to* 2*. For any* $\epsilon > 0$*, choose a natural number* N *so that* $2^N > \frac{3}{\epsilon} - 1$*. If* $n \ge N$*, then* $2^n \ge 2^N$ *and we have*

$$\left|\left(2 + \frac{3}{2^n+1}\right) - 2\right| = \frac{3}{2^n+1} \le \frac{3}{2^N+1} < \frac{3}{\frac{3}{\epsilon} - 1 + 1} = \epsilon.$$

This proves that the limit is equal to 2*.*

3. For each of the following series, state whether it is convergent or divergent. Justify your answer with an appeal to an appropriate convergence test or theorem. If convergent, give the value of the sum.

(a) $\sum_{i=0}^{\infty} 5(\frac{1}{2})^i$

This is a convergent geometric series of the form $\sum_{i=0}^{\infty} ar^i$ *with* $|r| < 1$*. The value of the sum is* $\frac{a}{1-r} = \frac{5}{1-1/2} = 10$.

(b) $\sum_{i=1}^{\infty}\frac{1}{2i}$

This is a divergent series. It is equal to $\frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{i}$*, and the harmonic series* $\sum_{i=1}^{\infty}\frac{1}{i}$ *diverges.*

(c) $\sum_{i=2}^{\infty}\left(\frac{1}{i^2}-\frac{1}{(i+1)^2}\right)$

This is a convergent telescoping series. The partial sum

$$S_N=\sum_{i=2}^{N}\left(\frac{1}{i^2}-\frac{1}{(i+1)^2}\right)=(\frac{1}{4}-\frac{1}{9})+(\frac{1}{9}-\frac{1}{16})+\cdots+(\frac{1}{N^2}-\frac{1}{(N+1)^2})=\frac{1}{4}-\frac{1}{(N+1)^2}.$$

Since $0<\frac{1}{(N+1)^2}<\frac{1}{N^2}$ *and* $\lim_{N\to\infty}\frac{1}{N^2}=0$ *(as discussed in class),* $\lim_{N\to\infty}\frac{1}{(N+1)^2}=0$ *by the Squeeze Theorem. Therefore,* $\sum_{i=2}^{\infty}\left(\frac{1}{i^2}-\frac{1}{(i+1)^2}\right)=\lim_{N\to\infty}\frac{1}{4}-\frac{1}{(N+1)^2}=\frac{1}{4}$.

(d) $\sum_{i=1}^{\infty}\frac{2i+1}{i}$

This is a divergent series, since $\lim_{i\to\infty}\frac{2i+1}{i}=\lim_{i\to\infty}(2+\frac{1}{i})=2+\lim_{i\to\infty}\frac{1}{i}=2\neq 0$.

4. (a) Let $f(x)=\frac{2x+1}{x}$. Find a formula for $f^{-1}(x)$. $\boxed{f^{-1}(x)=\frac{1}{x-2}}$

(b) Let g, h be two functions satisfying $g(1)=1$, $g(2)=3$, $g(3)=2$, and $h(1)=2$, $h(2)=3$, $h(3)=5$. Let $f(x)=g(h(x))$. For each of the following expressions, find the value or explain clearly why it is impossible to find the value based on the given information: (i) $f(1)$, (ii) $f(3)$

$f(1)=g(h(1))=g(2)=3$. $f(3)=g(h(3))=g(5)$ *cannot be determined from given information.*

5. A ball is propelled upwards from ground level with an initial velocity of V meters/second. The formula for the height $h(t)$ of the ball after t seconds is $h(t)=Vt-\frac{1}{2}gt^2$, where g is the acceleration due to gravity. The ball reaches the ground again at time $T=\frac{2V}{g}$. It reaches its maximum height at time $\frac{1}{2}T=\frac{V}{g}$; the maximum height reached at this time is $h(\frac{V}{g})=\frac{V^2}{2g}$. When the ball strikes the ground again at time T, its downward velocity is again V meters/second.

(a) After striking the ground, the ball rebounds with a new upward velocity equal to a certain fraction of the downward velocity immediately before impact. Assume that the upward velocity v_n immediately after the nth bounce satisfies $v_n=\frac{1}{2}v_{n-1}$ for each $n\geq 1$. Find a formula for v_n in terms of n and $v_0=V$.

The first few velocities are $v_1=\frac{1}{2}v_0$, $v_2=\frac{1}{2}v_1=\frac{1}{4}v_0$, $v_3=\frac{1}{2}v_2=\frac{1}{8}v_0$, *etc. In general,* $v_n=(\frac{1}{2})^n v_0=2^{-n}v_0$.

(b) Find the maximum height h_n attained between the nth and $(n+1)$st bounces of the ball, and the time t_n between these bounces. Your answers should be given in terms of n, v_0 and g.

By the information given above, $t_n=\frac{2v_n}{g}=2^{1-n}\frac{v_0}{g}$. *Also* $h_n=h(t_n)=\frac{v_n^2}{2g}=\frac{(2^{-n}v_0)^2}{2g}=4^{-n}\frac{v_0^2}{2g}$.

(c) Calculate the total time elapsed until the ball comes to rest, and the total distance travelled by the ball. Both answers should be expressed entirely in terms of v_0 and g. Draw a sketch to illustrate the complete path of the ball.

The total time elapsed until the ball comes to rest is $\sum_{n=0}^{\infty}t_n=\sum_{n=0}^{\infty}2^{1-n}\frac{v_0}{g}=\frac{2v_0}{g}\frac{1}{1-1/2}=\frac{4v_0}{g}$. *The total distance travelled by the ball is* $\sum_{n=0}^{\infty}2h_n$ *(during each bounce, the ball must go up and down!)* $=\sum_{n=0}^{\infty}4^{-n}\frac{v_0^2}{g}=\frac{v_0^2}{g}\frac{1}{1-1/4}=\frac{4v_0^2}{3g}$.