Math 221 Quiz #11 Solutions

1. A rectangular sheet of cardboard of area $48\,\text{in}^3$ is to be dissected by four vertical cuts into five pieces (one square and four congruent rectangles) which are then arranged to form a box with a square base and an open top. Find the dimensions of the rectangle which produce the box with largest volume.

Answer: Let a be the width of the sheet of cardboard and let b be the length (in inches).

Cutting the sheet of cardboard as indicated produces a square piece of dimensions $a \times a$ and four congruent rectangular pieces of dimensions $a \times c$, where $a + 4c = b$. (See the figure.) Arrange these into a box of dimensions $a \times a \times c$. The volume of the box is $V = a^2 c\,\text{in}^3$. The area of the original sheet is $ab = 48\,\text{in}^2$. The problem becomes

Maximize $V = a^2 c$ subject to the constraints $a + 4c = b$ and $ab = 48$.

Eliminating b from the constraints gives $a + 4c = \frac{48}{a}$ or

$$c = \frac{1}{4}\left(\frac{48}{a} - a\right) = \frac{12}{a} - \frac{a}{4}.$$

Note that $a > 0$ (since it is a unit of length) and also $a < b$ (since $c > 0$ is also a unit of length); since $ab = 48$ this means $a^2 < ab = 48$ or $a < \sqrt{48}$.

We will maximize

$$V(a) = a^2 c = a^2\left(\frac{12}{a} - \frac{a}{4}\right) = 12a - \frac{1}{4}a^3$$

over the interval $[0, \sqrt{48}]$. Note that $V(0) = V(\sqrt{48}) = 0$. Compute

$$V'(a) = 12 - \frac{3}{4}a^2.$$

We have $V'(a) = 0$ if and only if $a^2 = 16$ or $a = 4$ (the case $a = -4$ is physically unrealistic). Note that if $a = 4$, then $b = 12$ and $c = 2$.

The required rectangle has dimensions $4\,\text{in} \times 12\,\text{in}$ producing a box of dimensions $4\,\text{in} \times 4\,\text{in} \times 2\,\text{in}$ and volume $32\,\text{in}^3$.