Math 221 Quiz #9 Solutions

1. (a) Give a careful and complete statement of Rolle's Theorem, including all hypotheses and the conclusion.

   Answer: Assume that f is a function which is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b) = 0$ then there exists c, $a < c < b$, so that $f'(c) = 0$.

(b) Show that the function $f(x) = x^3 - 2x^2$ satisfies the assumptions of Rolle's theorem on $[0, 2]$. Find a value of c which makes the conclusion of Rolle's theorem true. Illustrate with a sketch.

Answer: f is clearly continuous on $[0, 2]$ and differentiable on $(0, 2)$. Also $f(0) = 0$ and $f(2) = 8 - 2 \cdot 4 = 0$. According to Rolle's theorem, there exists c, $0 < c < 2$, so that

$$0 = f'(c) = 3c^2 - 4c.$$

Thus either $c = 0$ or $c = \frac{4}{3}$; since the first one of these is ruled out by assumption ($0 < c < 2$) we choose $c = \frac{4}{3}$.

(c) Let $f(x) = x^4 + 2x^2 - 5$. Prove that there are exactly two distinct values x for which $f(x) = 0$.

Answer: The Fundamental Theorem of Algebra ensures that there are at most four zeros of f. Since $f(0) = -5$ and $f(\pm 2) = 16 + 8 - 5 = 19$ there are zeros of the derivative f' in the intervals $(-2, 0)$ and $(0, 2)$. Thus f has at least two zeros. To see that it has at most two zeros, note that between every two zeros of a function g there corresponds a zero of the derivative. Thus

$$\text{number of distinct real zeros of } f \leq \text{number of distinct real zeros of } f'.$$

It suffices then to show that f' has at most one real zero. Compute

$$f'(x) = 4x^3 + 4x = 4x(x^2 + 1).$$

This has only one real zero, at $x = 0$. Thus f has at most two zeros. Putting it all together, f has exactly two zeros.