Math 221 Recitation Exercises 1

Problems marked with a (*) are more challenging. Try these only after you've completed all of the other problems!

1. Calculate $\sqrt{2}$ accurate to eight decimal places (calculator allowed).

2. Prove the identity
$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$$
for all real numbers m and n. Explain how this identity can be used to construct Pythagorean triples, i.e., triples of positive integers which form the sides of a right triangle. Find a choice of m and n which gives the $3-4-5$ triangle.

3. Show that if n is a natural number and n^3 is even, then n^3 is a multiple of 8 and n is even.

4. (a) Define a sequence of numbers $x_0, x_1, x_2, \ldots$ by the following recursive scheme:
$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) \qquad \text{for } n = 0, 1, 2, 3, \ldots \tag{1}$$
For example, if $x_0 = 2$, then $x_1 = x_{0+1} = \frac{1}{2}(2 + \frac{2}{2}) = \frac{3}{2} = 1.5$ (apply (1) with $n = 0$), $x_2 = x_{1+1} = \frac{1}{2}(\frac{3}{2} + \frac{2}{3/2}) = \frac{17}{12} = 1.41666\ldots$ (apply (1) with $n = 1$), and so on.

   Pick several different **positive** values for x_0 and calculate the first few terms in the sequence $x_0, x_1, x_2, \ldots$ for each choice. What happens to the sequence? Can you make a conjecture about what happens for all $x_0 > 0$? How quickly does the sequence approach the limiting value?

   (b) Find all positive solutions x to the equation $x = \frac{1}{2}(x + \frac{2}{x})$.

5. Let $f(x) = x^2 - 2$. Compute the expression
$$x - \frac{f(x)}{f'(x)}$$
and simplify. (**NOTE:** $f'(x)$ denotes the derivative of $f(x)$ with respect to x.) Show that the recursive scheme
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2}$$
coincides with the scheme in (1).

   Remark: You might recognize the recursion (2) as *Newton's method* for finding zeros of the function $f(x)$. We'll discuss Newton's method in more detail later in the course.

6. (*) Adapt the argument which we gave in class to show that $\sqrt{n}$ is irrational whenever n is a prime number. What goes wrong with the argument if we choose $n = 4$? How should the argument be modified to deal with the case when n is a composite (non-prime) integer which is not a perfect square (for example, $n = 6$)?