

Math 221 Recitation Exercises 2

1. Let $f(x) = \frac{1}{2}x + 3$. Define a sequence x_0, x_1, x_2, \dots by the recursion $x_{n+1} = f(x_n)$.

- (a) Pick several initial values for x_0 and compute the first few terms in the sequence.
- (b) Does $\lim_{n \rightarrow \infty} x_n$ exist for every choice of x_0 ? If so, what does it equal? Justify your answer. (Hint: rewrite the formula for f in the form $f(x) = a + \frac{1}{2}(x - a)$ for some a .)
- (c) Show that f is 1-1 and calculate $f^{-1}(y)$.
- (d) Explain why part (c) implies that for each $x_0 \in \mathbb{R}$ we can find a unique two-sided sequence

$$\dots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, x_2, x_3, \dots$$

so that $x_{n+1} = f(x_n)$ for **every** integer n .

We can represent the situation in part (d) pictorially as follows:

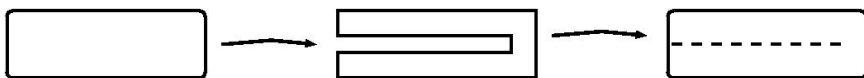
$$\dots \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \bullet \xrightarrow{f} \dots \quad (1)$$

2. Let $g(x) = 4x(1 - x)$. Define a sequence x_0, x_1, x_2, \dots by the recursion $x_{n+1} = g(x_n)$.

- (a) Pick several initial values for x_0 satisfying $0 < x_0 < 1$ and compute the first few terms in the sequence.
- (b) What, if anything, can you say about the long-term behavior of the sequence (x_n) ? Does $\lim_{n \rightarrow \infty} x_n$ necessarily exist? Does this limit ever exist? You do not need to give a precise mathematical justification, but should give some evidence for your claims.
- (c) Solve $y = 4x(1 - x)$ for x in terms of y .
- (d) Explain why part (c) implies that for each $x_0 \in (0, 1)$ there are two different values $a, b \in (0, 1)$ so that $g(a) = g(b) = x_0$. Illustrate with a diagram similar to (1).

3. (**Odometer**) The odometer on a certain model of car registers only five digits. In other words, when the car goes from its 99,999th mile to its 100,000th mile the odometer changes from 99999 to 00000. Consider the function which maps a given car of this model to its odometer reading. Is this function 1-1? Is this function onto?

4. (**Kneading**) The *baker's transformation* (kneading a loaf of bread) is the following transformation of a set S :



Consider the baker's transformation as a function $h : S \rightarrow S$, and define a sequence of points (P_n) by $P_{n+1} = h(P_n)$. Describe how the dynamics of this sequence are similar to or different from that in problems #1 and #2.