Math 221 Recitation Exercises 3

Let $y = f(x)$ be a function defined for x in the interval $[a, b]$. Fix a positive integer n. For each $i = 0, 1, 2, \ldots, n$, define

$$x_i = a + (b-a)\frac{i}{n}$$

and

$$\Delta x_i = x_{i+1} - x_i = \left[a + (b-a)\frac{i+1}{n}\right] - \left[a + (b-a)\frac{i}{n}\right] = \frac{b-a}{n}.$$

For each $i = 1, 2, \ldots, n$ choose a value c_i in the interval $[x_{i-1}, x_i]$.

The *Riemann sum* for this data is

$$\sum_{i=1}^{n} f(c_i)\Delta x_i.$$

Examples: (1) The *left Riemann sum* is obtained by choosing $c_i = x_{i-1}$. It is equal to

$$\sum_{i=1}^{n} f(x_{i-1})\Delta x_i.$$

(2) The *right Riemann sum* is obtained by choosing $c_i = x_i$. It is equal to

$$\sum_{i=1}^{n} f(x_i)\Delta x_i.$$

(3) The *lower Riemann sum* is obtained by choosing a value c_i for which

$$f(c_i) = \min\{f(x) : x_{i-1} \le x \le x_i\},$$

i.e., f takes on its minimum value on the interval $[x_{i-1}, x_i]$ at $x = c_i$. It is equal to

$$\sum_{i=1}^{n} \left[\min_{x_{i-1} \le x \le x_i} f(x)\right] \Delta x_i.$$

(At least one such c_i exists by the Extreme Value Theorem. It may not be easy to figure out exactly what its value is! There may be more than one point in $[x_{i-1}, x_i]$ where f takes on its minimum, but the **minimum value** $f(c_i)$ is the same for any such point, so the Riemann sum is well-defined.)

(4) The *upper Riemann sum* is obtained by choosing a value c_i for which

$$f(c_i) = \max\{f(x) : x_{i-1} \le x \le x_i\},$$

i.e., f takes on its maximum value on the interval $[x_{i-1}, x_i]$ at $x = c_i$. It is equal to

$$\sum_{i=1}^{n} \left[\max_{x_{i-1} \le x \le x_i} f(x)\right] \Delta x_i.$$

Definition 1. *The definite integral of $y = f(x)$ from $x = a$ to $x = b$ is the limit as $n \to \infty$ of any such sequence of Riemann sums, defined with any choice of points c_i. Note: the limit may not exist! If it does, we say that f is Riemann integrable on this interval. Not all functions are Riemann integrable. We write $\int_a^b f(x)\,dx$ for the definite integral of $y = f(x)$ from $x = a$ to $x = b$.*

If $f(x) > 0$ for $a \le x \le b$, $\int_a^b f(x)\,dx$ computes the area of the region bounded by the graph of f, the x-axis and the lines $x = a$ and $x = b$. For general functions f, $\int_a^b f(x)\,dx$ computes the *signed area* between the graph of f and the x-axis between $x = a$ and $x = b$, as discussed in class.

Exercises

In the first six problems below, let $y = f(x) = 3x + 1$ be defined on the interval $1 \leq x \leq 3$.

(1) For a fixed but arbitrary natural number n, find formulas for x_i and Δx_i.

(2) Write out the expression for the right Riemann sum $\sum_{i=1}^{n} f(x_i)\Delta x_i$.

(3) Evaluate the expression you found in the previous problem. You may need to use the formulas $\sum_{i=1}^{n} 1 = n$ and $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. Express your answer in the form $A + \frac{B}{n}$ for some explicit values A and B.

(4) Show that the answer to the previous problem has a limit as $n \to \infty$ and compute it.

(5) Sketch the graph of the function $y = f(x) = 3x + 1$ on the interval $1 \le x \le 3$ and the region bounded by the graph of f, the x-axis, and the lines $x = 1$ and $x = 3$. Use elementary plane geometry to find the area of this region. Compare with your answer to the previous problem.

(6) **Uses calculus!** Use the Fundamental Theorem of Calculus to find $\int_1^3 (3x + 1)\,dx$. Compare your answer to the previous two problems.

In these problems, let $y = f(x)$ be an arbitrary function defined on an arbitrary interval $a \leq x \leq b$.

(7) **Midpoint Riemann sums.** Let c_i be the midpoint of $[x_{i-1}, x_i]$, where x_i is as on the first page. Find a formula for c_i and write the corresponding Riemann sum.

(8) **Trapezoid Rule.** Consider the average of the left and right Riemann sums

$$\frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\Delta x_i + \sum_{i=1}^{n} f(x_i)\Delta x_i\right] = \sum_{i=1}^{n}\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right)\Delta x_i.$$

Explain why this expression is different from the expression you wrote in the previous problem, for general functions f. For which functions is it the same? Draw a picture to illustrate what this expression is computing.

(Hint: the quantity $\left(\frac{f(x_{i-1})+f(x_i)}{2}\right)\Delta x_i$ calculates the area of a *trapezoid* with two parallel vertical sides of length $f(x_{i-1})$ and $f(x_i)$ at a horizontal distance Δx_i from each other.)