Math 221 Recitation Exercises 6
Derivatives

(1) Use a calculator to plot $y = f(x) = a^x$ as well as $y = \frac{a^{x+h} - a^x}{h}$ for the choices $h = 1$, $h = 0.5$, $h = 0.2$ and $h = 0.1$ in each of the following cases. Use the window $-4 \le x \le 4$, $0 \le y \le 4$.
 (a) $a = 2$
 (b) $a = 3$
 (c) $a = 2.7$

(2) Consider the function $f(x) = a^x$ for $a > 0$. Show that

$$\frac{f(x+h) - f(x)}{h} = c_1(h, a)f(x)$$

for each $h \neq 0$, where $c_1(h, a)$ is a value that depends only on h and a. Then show that

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = c_2(a)f(x),$$

where $c_2(a)$ is a value that depends only on a. Find a formula for $c_2(a)$ in terms of $c_1(h, a)$.

The base of the natural logarithm is $e = \lim_{n \to \infty}(1 + \frac{1}{n})^n \approx 2.718281828\ldots$. Problem 1 suggests the following result, which can be rigorously proven:

Theorem 1. *If $f(x) = e^x$, then $f'(x) = e^x$.*

(3) Find the derivatives of each of the following:

 (a) $f(x) = x^n e^x$ for an arbitrary positive integer n.

 (b) $f(x) = \frac{1}{2}(e^x + e^{-x})$ and $g(x) = \frac{1}{2}(e^x - e^{-x})$.

 (c) $f(x) = e^{g(x)}$. Your answer should involve $g(x)$ and $g'(x)$.

Remark 2. *The hyperbolic cosine function is* $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$*. The hyperbolic sine function is* $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$*.*

(4) Use a calculator to plot $y = \sin(x)$ as well as $y = \frac{\sin(x+h)-\sin(x)}{h}$ for the choices $h = 1$, $h = 0.5$, $h = 0.2$ and $h = 0.1$. Use the window $-10 \leq x \leq 10$, $-2 \leq y \leq 2$. What does your graph suggest about the value of the derivative of $y = \sin(x)$?

Problem 4 suggests part (a) of the following result, which can be rigorously proven:

Theorem 3. *(a) If* $f(x) = \sin(x)$*, then* $f'(x) = \cos(x)$*. (b) If* $g(x) = \cos(x)$*, then* $g'(x) = -\sin(x)$*.*

Proof of (b). We use the trig identity $g(x) = \cos(x) = \sin(\frac{\pi}{2} - x) = f(\frac{\pi}{2} - x)$.

By the Chain Rule,

$$g'(x) = f'(\frac{\pi}{2} - x) \cdot (-1) = -\cos(\frac{\pi}{2} - x) = -\sin\left(\frac{\pi}{2} - (\frac{\pi}{2} - x)\right) = -\sin(x).$$

□

(5) Find the derivatives of each of the following using only differentiation rules which have already been discussed in lecture or recitation.

(a) $f(x) = \frac{\sin(x)}{\cos(x)}$.

(b) $f(x) = e^{ax}\sin(bx)$ for real numbers a and b.

(c) $f(x) = \cos\left(\frac{x^2+1}{\sin(x)}\right)$.