Math 221 Recitation Exercises 7
Implicit Differentiation

Let $F(x, y)$ be a function with two independent variables x and y.

Examples 1. $F(x,y) = x^2 + y^3 - 1$, $G(x,y) = x^3 + y^3 - 6xy$, $H(x,y) = x\cos(y) + e^{xy} - y^2$.

The *partial derivatives* of F with respect to x and y are the functions $\frac{\partial F}{\partial x}(x,y)$ and $\frac{\partial F}{\partial y}(x,y)$ defined by differentiating F with respect to either x or y, while holding the other variable constant.

Examples 2. *(1)* $F(x,y) = x^2 + y^3 - 1$, $\frac{\partial F}{\partial x}(x,y) = 2x$, $\frac{\partial F}{\partial y}(x,y) = 3y^2$.

(2) $G(x,y) = x^3 + y^3 - 6xy$, $\frac{\partial G}{\partial x}(x,y) = 3x^2 - 6y$ $\frac{\partial G}{\partial y}(x,y) = 3y^2 - 6x$

(3) $H(x,y) = x\cos(y) + e^{xy} - y^2$, $\frac{\partial H}{\partial x}(x,y) = \cos(y) + ye^{xy}$ $\frac{\partial H}{\partial y}(x,y) = -x\sin(y) + xe^{xy} - 2y$.

We may or may not be able to solve the equation $F(x,y) = 0$ for y as an explicit function of x.

Examples 3. *(1)* $F(x,y) = x^2 + y^3 - 1 = 0$ *has the explicit solution* $y = (1-x^2)^{1/3}$.

(2) $G(x,y) = x^3 + y^3 - 6xy = 0$ *has an explicit solution (which can be found by using the cubic equation), but it is pretty complicated.*[1]

(3) $H(x,y) = x\cos(y) + e^{xy} - y^2 = 0$ *has no closed-form explicit solution.*

When F is a sufficiently nice function, the equation $F(x,y) = 0$ defines a smooth curve in the plane. Implicit differentiation is a method for determining the tangent lines to this curve, without explicitly solving for $y = y(x)$ in terms of x.

Implicit Differentiation. The *total derivative* of $F(x, y(x))$ with respect to x is

$$\frac{dF}{dx} = \frac{d}{dx}F(x,y(x)) = \frac{\partial F}{\partial x}(x,y) + \frac{\partial F}{\partial y}(x,y)\cdot\frac{dy}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot y'.$$

Using this, the derivative y' at a point (x_0, y_0) on the curve defined by $F(x,y) = 0$ can be computed as

$$y'|_{(x_0,y_0)} = -\frac{(\partial F/\partial x)(x_0,y_0)}{(\partial F/\partial y)(x_0,y_0)}.$$

WARNING! Be careful to distinguish the **partial derivative** and the **total derivative** of F with respect to x. In one, we treat x and y as independent variables. In the other, we treat $y = y(x)$ as a function of x.

Examples 4. *(1)* $F(x,y) = x^2+y^3-1$, $\frac{d}{dx}F(x,y(x)) = 2x+3y^2y'$ *so* $F(x,y) = 0$ *implies* $y' = -\frac{2x}{3y^2}$.

(2) $G(x,y) = x^3 + y^3 - 6xy$, $\frac{d}{dx}G(x,y(x)) = (3x^2 - 6y) + (3y^2 - 6x)y'$ *so* $G(x,y) = 0$ *implies* $y' = -\frac{3x^2-6y}{3y^2-6x}$.

(3) $H(x,y) = x\cos(y) + e^{xy} - y^2$, $\frac{dH}{dx} = (\cos(y) + ye^{xy}) + (-x\sin(y) + xe^{xy} - 2y)y'$ *so* $H(x,y) = 0$ *implies*

$$y' = -\frac{\cos(y) + ye^{xy}}{-x\sin(y) + xe^{xy} - 2y}.$$

[1] In fact, the equation $x^3 + y^3 - 6xy = 0$ defines three different explicit functions $y = f(x)$, the simplest one of which is $y = \left(-\frac{1}{2}x^3 + \sqrt{\frac{1}{4}x^6 - 8x^3}\right)^{1/3} + \left(-\frac{1}{2}x^3 - \sqrt{\frac{1}{4}x^6 - 8x^3}\right)^{1/3}$!

Exercises

(1) Find y' if $\sin(x+y) = y^2\cos(x)$. Hint: Let $F(x,y) = \sin(x+y) - y^2\cos(x)$.

(2) Find y' if $x = \sin(y)$. Express your answer in terms of x only, and use this answer to find the derivative of $y = \arcsin(x)$. Repeat for $x = \cos(y)$ and $y = \arccos(x)$.

(3) An *elliptic curve* is any curve defined by an equation of the form $y^2 = p(x)$, where $p(x)$ is a cubic polynomial. In this problem, we work with the elliptic curve $y^2 = x^3 + x^2 - 6x$.

(a) Find y' for this elliptic curve.

(b) Verify that $(3, 3\sqrt{2})$ is a point on this curve. Find the value of y' at $(3, 3\sqrt{2})$. Write the equation of the tangent line at this point.

(c) Find all points (x_0, y_0) where the slope of the tangent line $y'|_{(x_0,y_0)}$ is undefined.

(d) Find the x-coordinates of each point where the slope of the tangent line is zero.

(e) Plot $y^2 = x^3 + x^2 - 6x$ on your calculator. (Hint: Plot $y = \sqrt{x^3 + x^2 - 6x}$ and $y = -\sqrt{x^3 + x^2 - 6x}$ on the same set of axes.) Confirm your answers to parts (b),(c) and (d).

(4) The *Bessel function of order zero*, $y = J_0(x)$, is a solution to the differential equation $xy'' + y' + xy = 0$ satisfying $J_0(0) = 1$. In other words, $xJ_0''(x) + J_0'(x) + xJ_0(x) = 0$ for all x. Find the value of $J_0'(0)$. Then use implicit differentiation to find the value of $J_0''(0)$.