Math 221 Recitation Exercises 8
Using the derivative and second derivative to sketch graphs

Let $y = f(x)$ be a continuous function defined for x in a closed interval $[a, b]$.

A point x_0 is a *critical point* of f if $f'(x_0) = 0$ or $f'(x_0)$ is undefined.

Let (c, d) be an open interval contained in $[a, b]$. If f is differentiable on (c, d), then f is increasing on (c, d) if $f' > 0$ on (c, d), while f is decreasing on (c, d) if $f' < 0$ on (c, d).

Now assume that f is twice differentiable on (c, d) (i.e., $f''(x)$ exists for all x in (c, d)). Then f is concave up on (c, d) if $f'' > 0$ on (c, d), while f is concave down on (c, d) if $f'' < 0$ on (c, d). A point x_0 is an *inflection point* for f if the concavity of f changes at x_0, i.e., if f'' changes sign at x_0.

(1) Find all critical points and points of inflection for the function $y = f(x) = x^4 - 4x^3$. Find the value of f at each of these points. Find all intervals on which f is (i) increasing, (ii) decreasing, (iii) concave up, (iv) concave down. Use this information to sketch the graph of f.

(2) In this problem, we investigate a family of graphs depending on an additional parameter. For any real number c, consider the function

$$f_c(x) = \frac{1}{x^2 + 2x + c}.$$

(We include c as a subscript to distinguish these functions for different values of the parameter.) Note: your answers to any of the following parts may depend on c!

(a) What is the domain of definition for f_c? Give equations for all horizontal asymptotes (HA) and vertical asymptotes (VA) for f_c. For which choices of c are there two VA? For which c is there one VA? For which c are there no VA?

(b) For $f_c(x) = \frac{1}{x^2+2x+c}$, compute $f_c'(x)$ and find all critical points of f_c which lie in the domain of definition. (Hint: be careful about the case $c = 1$!) Find the value of f_c at each of these points.

(c) Find all intervals on which f_c is increasing. Find all intervals on which f_c is decreasing.

(d) Compute $f_c''(x)$ and find all inflection points for f_c in its domain of definition. Find the value of f_c at each of these points. Find all intervals on which f_c is concave up/concave down.

(e) Give a rough sketch of the graph of f_c in three cases: $c = 0$, $c = 1$ and $c = 2$.