Fourier series exercise

Suppose that $f(t)$ is a P-periodic function. Set $L = \frac{P}{2}$ to be the *half-period* of the function f. The *Fourier series* for $f(t)$ is the infinite series

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi t}{L}) + \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi t}{L}), \tag{1}$$

where the coefficients A_n, $n = 0, 1, 2, \ldots$, and B_n, $n = 1, 2, \ldots$ are given by the expressions

$$A_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi t}{L})\,dt$$

and

$$B_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin(\frac{n\pi t}{L})\,dt.$$

The following theorem gives conditions under which the Fourier series (1) converges and indicates the value to which it converges. The hypotheses of the theorem are not optimal — Fourier series can converge for functions which have worse discontinuities than the type described in the theorem — but this theorem will cover all of the examples which we will look at in this course.

Theorem 1. *Suppose that f and f' are piecewise continuous, i.e., they have at most a finite number of finite jump discontinuities on any bounded interval. Then the Fourier series (1) for f converges for all values of t. The series converges*

(a) to $f(t)$ at each point t where f is continuous;

(b) to

$$\frac{\lim_{s\to t^-} f(s) + \lim_{s\to t^+} f(s)}{2}$$

at each point t where f is discontinuous.

Fourier series can be used to evaluate certain infinite sums. Given a function f satisfying the conditions in the theorem, we can write

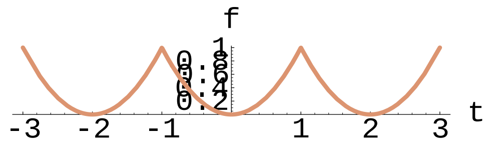
$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi t}{L}) + \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi t}{L}) = f(t)$$

at each point t where f is continuous and

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(\frac{n\pi t}{L}) + \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi t}{L}) = \frac{1}{2}\left(\lim_{s\to t^-} f(s) + \lim_{s\to t^+} f(s)\right)$$

at each point t where f is discontinuous. By specializing to specific values of t, we derive explicit formulas for certain infinite sums. The following exercise leads you through an example.

Consider the function f defined by setting $f(t) = t^2$ for $-1 \leq t \leq 1$ and extending 2-periodically.



In this case the half-period is $L = P/2 = 1$. Since $f(t)$ is an even function, the Fourier series will involve only cosine terms and will take the form

$$f(t) \sim \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi t).$$

Calculate the coefficients A_n and write out the first few terms in the Fourier series for f.[1]

[1] You will need to use the following indefinite integral $\int u^2 \cos u \, du = u^2 \sin u + 2u \cos u - 2 \sin u + C$.

If you calculated correctly on the previous page, you found

$$f(t) \sim \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi t).$$

Remember that $f(t) = t^2$ for $-1 \le t \le 1$. The periodic extension is continuous for all values of t, so we can write

$$f(t) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi t) \tag{2}$$

where the series converges for all $-1 \le t \le 1$.

Choosing $t = 1$ in (2), we find

$$\begin{aligned} 1 &\overset{\text{why?}}{=} \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2} \\ &= \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}. \end{aligned}$$

Therefore

$$\left(\frac{\pi^2}{4}\right)\left(1 - \frac{1}{3}\right) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and so

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{3}$$

By making a different choice of t, calculate

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \boxed{} \tag{4}$$

In fact, there are two different values $0 \le t \le 1$ which you can use to compute the value of the sum in (4). Can you find both of them?

Add together equations (3) and (4) and divide by two to determine the value of

$$\sum_{n \text{ odd}} \frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \cdots = \boxed{} \tag{5}$$

Then subtract equation (4) from equation (3) and divide by two to determine the value of

$$\sum_{n \text{ even}} \frac{1}{n^2} = \frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots = \boxed{} \tag{6}$$

Question: can you think of another way to find the value of the sum in (6)?