Homogeneous Linear Equations of Higher Order — Summary

The general *homogeneous linear differential equation* takes the form

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=0. \quad (\S)$$

We always assume that the *coefficient functions* $a_0(x), \ldots, a_n(x)$ are continuous functions defined on some interval.

Solutions to (§) satisfy the important *principle of superposition*, which states that linear combinations of solutions are again solutions:

> If $y_1, \ldots, y_n$ solve (§) and $c_1, \ldots, c_n$ are constants, then $c_1y_1 + \cdots + c_ny_n$ also solves (§).

Existence and Uniqueness Theorem: If $a_0(x), \ldots, a_{n-1}(x)$ are continuous functions on an interval containing a point x_0 and $b_0, \ldots, b_{n-1}$ are constants, then the initial value problem

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=0,$$
$$y(x_0)=b_0, y'(x_0)=b_1, \ldots, y^{(n-1)}(x_0)=b_{n-1},$$

has a unique solution on that interval.

A collection of functions $f_1, \ldots, f_n$ is *linearly dependent* on an interval if there exist constants $c_1, \ldots, c_n$, **not all equal to zero**, so that $c_1f_1(x)+\cdots+c_nf_n(x)=0$ for every x in the interval. If $f_1, \ldots, f_n$ are not linearly dependent, we say that they are *linearly independent*.

An easy way to test whether a collection of functions is linearly independent is to use the Wronskian. The *Wronskian* of n functions $f_1, \ldots, f_n$ is the new function

$$W(f_1, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

(This symbol represents the *determinant* of the matrix with these entries; for a quick review of matrices and determinants see the next page of this handout.)

Theorem: Let $y_1, \ldots, y_n$ be solutions to a homogeneous linear differential equation

$$y^{(n)}+a_{n-1}(x)y^{(n-1)}+\cdots+a_1(x)y'+a_0(x)y=0 \quad (\dagger)$$

on an interval where the coefficient functions $a_0(x), \ldots, a_{n-1}(x)$ are continuous. Then $W(x)$ is either identically equal to zero or never equal to zero throughout the interval and:

(i) if $W(y_1, \ldots, y_n)$ is never equal to zero, then $y_1, \ldots, y_n$ are linearly independent. In this case every solution Y to (†) can be written as a linear combination of the functions $y_1, \ldots, y_n$:

$$Y = c_1y_1 + \cdots + c_ny_n$$

for some constants $c_1, \ldots, c_n$;

(ii) if $W(y_1, \ldots, y_n)$ is identically equal to zero, then $y_1, \ldots, y_n$ are linearly dependent.

Constant Coefficient Equations

A *homogeneous constant coefficient equation* takes the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad (\ddagger)$$

for some real constants $a_0, \dots, a_n$.

To find the solutions of such an equation, we use the *characteristic equation*

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0. \qquad (\P)$$

The *Fundamental Theorem of Algebra* tells us that this equation always has n roots provided we include complex roots and count roots according to *multiplicity.*

1. Every real root r of (¶) corresponds to a solution $y(x) = e^{rx}$ to (‡).

2. If $r_1 = p + qi$ and $r_2 = p - qi$ are two roots of (¶) (occuring in a complex conjugate pair), then two linearly independent solutions to (‡) are $y_1(x) = e^{px}\cos(qx)$ and $y_2(x) = e^{px}\sin(qx)$.

3. If a root r of (¶) is repeated m times (or has multiplicity m), then m linearly independent solutions to (‡) are

$$y_1(x) = e^{rx}, \quad y_2(x) = xe^{rx}, \quad \dots, \quad y_m(x) = x^{m-1}e^{rx}.$$

Similarly, if a pair $r_1 = p + qi$ and $r_2 = p - qi$ or complex conjugate roots has multiplicity m, then $2m$ linearly independent solutions to (‡) are

$$\begin{gathered} y_1(x) = e^{px}\cos(qx), \quad y_2(x) = e^{px}\sin(qx), \\ \vdots \\ y_{2m-1}(x) = x^{m-1}e^{px}\cos(qx), \quad y_{2m}(x) = x^{m-1}e^{px}\sin(qx). \end{gathered}$$

All functions which arise in this manner are linearly independent. It follows that we can always write down a full set of n linearly independent solutions to (‡).

Review: Determinants of Matrices

The *determinant* is a function from the collection of square matrices to the real numbers. It is defined recursively. For a 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the determinant is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For a 3×3 matrix $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ the determinant can be computed by expanding along the top row:

$$\begin{aligned} \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} &= a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ &= a(ei - fh) - b(di - fg) + c(dh - eg) \\ &= aei + bfg + cdh - bdi - afh - ceg. \end{aligned}$$

Notice the alternating pattern of signs. In general, for an $n \times n$ matrix A whose (i, j)th component is a_{ij}, the determinant $|A|$ is

$$|A| = a_{11}|A_{11}| - a_{12}|A_{12}| + \cdots \pm a_{1n}|A_{1n}|, \qquad (\pounds)$$

where A_{ij} denotes the $(n-1) \times (n-1)$ matrix (the (i, j)*th minor of* A) obtained by removing the ith row and jth column of A. The determinants of the A_{ij} are computed recursively using (£).

It is not necessary to expand along the top row. Any particular row or column of A can be used. The signs which should be used for the terms in (£) come from the following pattern:

$$\begin{pmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$