Nonhomogeneous Linear Equations — Summary

The general *nonhomogeneous (forced) linear differential equation* is

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = F(x). \qquad (\S)$$

The general solution to this equation takes the form

$$\boxed{y(x) = y_c(x) + y_p(x)},$$

where y_c is the general solution to the *associated unforced equation*

$$a_n(x)y^{(n)} + \cdots + a_1(x)y' + a_0(x)y = 0 \qquad (\dagger)$$

and y_p is one particular solution to (§). There are two common methods for finding such a particular solution: *undetermined coefficients* and *variation of parameters*.

I. Method of Undetermined Coefficients: This works if $F(x)$ is a linear combination of any of the following functions or any products of these functions:

1. polynomials in x;

2. exponential functions e^{kx};

3. $\cos(kx)$ or $\sin(kx)$.

In this case, we guess that y_p will be a linear combination (with coefficients to be determined) of all terms in $F(x)$ and all terms which appear in their derivatives. Substituting this trial solution back into (§) will yield a collection of simultaneous linear equations for the coefficients, which can be solved to find y_p.

To illustrate we will find the general solution to the differential equation

$$y'' - 3y' + 2y = F(x)$$

for several choices of the forcing function $F(x)$.

Observe that the complementary solution is

$$y_c(x) = C_1e^x + C_2e^{2x}.$$

Examples:

(1) If $F(x) = x^3$ we try a solution of the form $\boxed{y_p(x) = Ax^3 + Bx^2 + Cx + D}$. Substituting into the differential equation and equating coefficients gives $A = 1/2$, $B = 9/4$, $C = 21/4$ and $D = 45/8$, so

$$y(x) = C_1e^x + C_2e^{2x} + \frac{1}{2}x^3 + \frac{9}{4}x^2 + \frac{21}{4}x + \frac{45}{8}.$$

(2) If $F(x) = x^2e^{3x}$ we try a solution of the form $\boxed{y_p(x) = Ax^2e^{3x} + Bxe^{3x} + Ce^{3x}}$. Substituting into the equation gives $A = 1/2$, $B = -3/2$ and $C = 7/4$, so

$$y_p(x) = C_1e^x + C_2e^{2x} + (\frac{1}{2}x^2 - \frac{3}{2}x + \frac{7}{4})e^{3x}.$$

(3) If $F(x) = \sin(2x)$ we try a solution of the form $\boxed{y_p(x) = A\cos(2x) + B\sin(2x)}$. Substituting into the equation gives $A = 3/20$ and $B = -1/20$, so

$$y_p(x) = C_1e^x + C_2e^{2x} + \frac{3}{20}\cos(2x) - \frac{1}{20}\sin(2x).$$

The procedure is slightly different in the case when one of the terms in $F(x)$ is a solution to the homogeneous equation (†). In this case, multiply the appropriate terms in the trial solution by the smallest power of x needed to eliminate duplication with terms in y_c.

Examples:

(1) For the differential equation

$$y'' - 3y' + 2y = e^x,$$

we observe that the forcing term $f(x) = e^x$ is duplicated in the complementary solution. Therefore, instead of choosing $y_p(x) = Ae^x$ for the trial solution, we try $\boxed{y_p(x) = Axe^x}$. Substituting into the equation and equating coefficients gives $A = -1$. The general solution is

$$y_p(x) = C_1e^x + C_2e^{2x} - xe^x.$$

(2) For the differential equation

$$y'' - 2y' + y = e^x,$$

the complementary solution is $y_c(x) = C_1e^x + C_2xe^x$. We observe that the forcing term $f(x) = e^x$ is duplicated in the complementary solution. In this case, we must use the trial solution $y_p(x) = Ax^2e^x$. Substituting into the original equation and equating coefficients gives $A = \frac{1}{2}$. The general solution is

$$y_p(x) = C_1e^x + C_2xe^x + \frac{1}{2}x^2e^x.$$

II. Method of Variation of Parameters: This method always works, but involves the computation of several integrals which in many cases cannot be done in closed form. For simplicity we only consider the case $n = 2$, in other words,

$$y'' + P(x)y' + Q(x)y = F(x). \tag{‡}$$

Let y_1 and y_2 be two linearly independent solutions to $y'' + P(x)y' + Q(x)y = 0$. Then a particular solution to (‡) is

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

where u_1 and u_2 satisfy the following two equations:

$$\begin{aligned} u_1'(x)y_1(x) + u_2'(x)y_2(x) &= 0, \\ u_1'(x)y_1'(x) + u_2'(x)y_2'(x) &= F(x). \end{aligned} \tag{£}$$

The solution to (£) is

$$\boxed{u_1'(x) = -\frac{y_2(x)F(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} = -\frac{y_2(x)F(x)}{W(x)}}$$

and

$$\boxed{u_2'(x) = \frac{y_1(x)F(x)}{y_1(x)y_2'(x) - y_1'(x)y_2(x)} = \frac{y_1(x)F(x)}{W(x)}},$$

where

$$W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

is the Wronskian of y_1 and y_2. To find u_1 and u_2 we just have to integrate these expressions for u_1' and u_2'; in many cases, however, these integrals are impossible to evaluate in closed form.

Consider the two examples from before.

Examples: (1) $y'' - 3y' + 2y = e^x$. Here

$$y_1(x) = e^x \text{ and } y_2(x) = e^{2x}$$

solve the associated homogeneous equation. The Wronskian of y_1 and y_2 is $W(x) = e^{3x}$. Hence

$$u_1'(x) = -\frac{(e^{2x})(e^x)}{e^{3x}} = -1, \qquad u_2'(x) = \frac{(e^x)(e^x)}{e^{3x}} = e^{-x}$$

so $u_1(x) = -x$, $u_2(x) = -e^{-x}$, and

$$\boxed{y_p(x) = (-x)(e^x) + (-e^{-x})(e^{2x}) = -e^x - xe^x}.$$

(2) $y'' - 2y' + y = e^x$. Here

$$y_1(x) = e^x \text{ and } y_2(x) = xe^x$$

are linearly independent solutions to the associated homogeneous equation. The Wronskian of y_1 and y_2 is $W(x) = e^{2x}$, whence

$$u_1'(x) = -\frac{(xe^x)(e^x)}{e^{2x}} = -x, \qquad u_2'(x) = \frac{(e^x)(e^x)}{e^{2x}} = 1,$$

$u_1(x) = -\frac{1}{2}x^2$, $u_2(x) = x$, and

$$\boxed{y_p(x) = (-\tfrac{1}{2}x^2)(e^x) + (x)(xe^x) = \tfrac{1}{2}x^2e^x}.$$

Notice that the particular solution in (2) is the same one which we found using the method of undetermined coefficients. This is not exactly the case in (1). Can you explain why the particular solutions which we found by these two methods are different and why this is not a contradiction?