Math 385 Spring 2007
Second Midterm Exam (Solutions)

1. Find the general solution to $y^{(4)} - 8y'' + 16y = 0$.

Answer: The characteristic equation $r^4 - 8r^2 + 16 = 0$ has roots $r = 2, 2, -2, -2$. The general solution is $\boxed{y(x) = C_1 e^{2x} + C_2 x e^{2x} + C_3 e^{-2x} + C_4 x e^{-2x}}$.

2. (a) Find the general solution to the equation $y'' - 2y' + y = e^{3x} + 2$.

Answer: The complementary equation $y'' - 2y' + y = 0$ has characteristic equation $r^2 - 2r + 1 = 0$ with roots $r = 1, 1$ and solution $y_c(x) = C_1 e^x + C_2 x e^x$. We guess a particular solution by the method of undetermined coefficients. Try a solution of the form $y_p(x) = Ae^{3x} + B$. Then $4Ae^{3x} + B = e^{3x} + 2$ so $A = \frac{1}{4}$, $B = 2$, and

$$\boxed{y(x) = C_1 e^x + C_2 x e^x + \tfrac{1}{4} e^{3x} + 2}$$

(b) Find the particular solution to this equation satisfying the initial conditions $y(0) = 3$ and $y'(0) = 2$.

Answer: Substitute $y(0) = 3$ and $y'(0) = 2$ into the solution from part (a) to find $C_1 = \frac{3}{4}$ and $C_2 = \frac{1}{2}$. The solution is

$$\boxed{y(x) = \tfrac{3}{4} e^x + \tfrac{1}{2} x e^x + \tfrac{1}{4} e^{3x} + 2}$$

3. (a) For which values of ω does resonance occur in the equation $x'' + 25x = 160\cos(\omega t) + 99\cos(2\omega t)$?

Answer: $\boxed{\omega = 5}$ and $\boxed{\omega = \frac{5}{2}}$.

(b) Solve the initial value problem $x'' + 25x = 160\cos(3t) + 99\cos(6t)$, $x(0) = 0$, $x'(0) = 0$.

Answer: The complementary solution is $x_c(t) = C_1\cos(5t) + C_2\sin(5t)$. Guess a particular solution of the form $x_p(t) = A\cos(3t) + B\cos(6t)$. We find $16A\cos(3t) - 11B\cos(6t) = 160\cos(3t) + 99\cos(6t)$ so $A = 10$ and $B = -9$. We find

$$x(t) = C_1\cos(5t) + C_2\sin(5t) + 10\cos(3t) - 9\cos(6t).$$

Substituting $x(0) = 0$ and $x'(0) = 0$ gives $C_1 = -1$ and $C_2 = 0$, so the answer is

$$\boxed{x(t) = 10\cos(3t) - \cos(5t) - 9\cos(6t)}$$

4. (a) Prove that $y_1(x) = e^x$, $y_2(x) = xe^x$ and $y_3(x) = e^{-x}$ are linearly independent on the real line.

Answer: One method is to calculate the Wronskian: $W(x) = W(y_1, y_2, y_3)(x) = 4e^x$ which is nonzero for any value of x. Another method is to analyze limiting behavior. If $C_1 y_1(x) + C_2 y_2(x) + C_3 y_3(x) =$

$C_1e^x + C_2xe^x + C_3e^{-x} = 0$ for all x, let $x \to -\infty$ to conclude $C_3 = 0$, set $x = 0$ to conclude $C_1 = -C_3 = 0$, then we have $C_2xe^x = 0$ for all x so $C_2 = 0$.

(b) Write a constant coefficient linear equation whose general solution is the general linear combination of the functions y_1, y_2, y_3 from part (a). You may give your answer using differential operators.

Answer: Roots of the characteristic equation are $r = 1, 1, -1$. The answer is $\boxed{(D - I)^2(D + I)y = 0}$ or $\boxed{y''' - y'' - y' + y = 0}$

(c) Suppose that $y(x) = C_1y_1(x) + C_2y_2(x) + C_3y_3(x)$ is a solution to your equation from part (b), where y_1, y_2, y_3 are the functions in part (a), and suppose that $\lim_{x\to+\infty} y(x) = 0$. What can you conclude about the constants C_1, C_2, and C_3?

Answer: If we **only** know that $\lim_{x\to+\infty} y(x) = 0$ then we can only conclude that $C_1 = C_2 = 0$, i.e., $y(x) = C_3y_3(x) = C_3e^{-x}$. We cannot say anything about the value of C_3 in this case; in fact, $C_3e^{-x} \to 0$ as $x \to \infty$ for any choice of C_3.

5. Let $f(t, x)$ be a function defined for $0 \le t \le 1$ and all x. Describe how to use Euler's method with step size $h = \frac{1}{100}$ to approximate the value of $x(1)$, where $x(t)$ is the solution to the differential equation $x' = f(t, x)$, $x(0) = 0$.

Answer: Let $t_0 = 0$ and $x_0 = 0$. Define $t_{n+1} = t_n + h$ and $x_{n+1} = x_n + hf(t_n, x_n)$ for $n = 1, 2, \ldots, 99$. Then $t_{100} = 1$ and x_{100} approximates the value of $x(1)$, where $x(t)$ solves the differential equation $x' = f(t, x)$, $x(0) = 0$.

6. The *Heaviside step function* is defined to be

$$\text{step}(x) = \begin{cases} +1 & x > 0, \\ 0 & x \le 0. \end{cases}$$

Find a solution $y(x)$ to the differential equation $y'' + \text{step}(x)y = 0$ so that y and y' are continuous for all values of x, and $y(0) = 1$, $y'(0) = 0$.

Answer: For $x > 0$ we have $\text{step}(x) = 1$ and the equation is $y'' + y = 0$ with general solution $y(x) = C_1\cos(x) + C_2\sin(x)$. For $x \le 0$ we have $\text{step}(x) = 0$ and the equation is $y'' = 0$ with general solution $y(x) = D_1 + D_2x$. We want a solution $y(x)$ so that y and y' are continuous for all x, $y(0) = 1$ and $y'(0) = 0$. Thus

$$D_1 = \lim_{x\to0^-} y(x) = 1 = \lim_{x\to0^+} y(x) = C_1$$

and

$$D_2 = \lim_{x\to0^-} y'(x) = 0 = \lim_{x\to0^+} y'(x) = C_2$$

The answer to the problem is

$$y(x) = \begin{cases} \cos(x) & x > 0, \\ 1 & x \le 0. \end{cases}$$