Math 385 Spring 2007
Extra problem for Homework #9

A positive function w defined on an interval $[a, b]$ is called a *weight*. We say that two functions f and g are *orthogonal on the interval* $[a, b]$ *with respect to the weight* $w(x)$ if

$$\int_a^b f(x)g(x)w(x)\,dx = 0.$$

(a) Show that the functions $f(x) = x$ and $g(x) = 2x^2 - 1$ are orthogonal on the interval $[-1, 1]$ with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$. (Hint: make the substitution $x = \cos\theta$ in the integral.)

(b) Repeat part (a) for the functions $f(x) = x$ and $h(x) = 4x^3 - 3x$. (Hint: the same substitution is still helpful.)

(c) Show that $g(\cos\theta) = \cos(2\theta)$ and $h(\cos\theta) = \cos(3\theta)$ for every θ.

(d) (EXTRA CREDIT) For each positive integer n, there is a polynomial T_n of degree n satisfying $T_n(\cos\theta) = \cos(n\theta)$ for all θ. Show that T_n and T_m are orthogonal on $[-1, 1]$ with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ if $m \neq n$. Calculate

$$\int_{-1}^1 T_n(x)^2 w(x)\,dx.$$

Part (c) shows that $T_2(x) = 2x^2 - 1$ and $T_3(x) = 4x^3 - 3x$. Calculate $T_4(x)$.

Remark. T_n is called the *nth Chebyshev polynomial of the first kind*. Chebyshev polynomials are used to find best approximating polynomials for non-polynomial functions. See

`mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html`

and

`mathworld.wolfram.com/ChebyshevApproximationFormula.html`

for more information.