Math 385 Spring 2007
Quiz 3

1. Find the solution to the initial value problem

$$y'' - 4y = 0, \qquad y(0) = a, y'(0) = b.$$

Here a and b are arbitrary (but fixed) real numbers; your answer should depend on a and b.

Answer: Method I (easier): Two linearly independent solutions to the equation are $y_1(x) = \cosh(2x)$ and $y_2(x) = \sinh(x)$. In the general solution $y(x) = C_1 \cosh(2x) + C_2 \sinh(2x)$ we substitute the initial conditions to find the system of linear equations

$$C_1 = a \qquad 2C_2 = b.$$

Thus the particular solution is

$$\boxed{y(x) = a\cosh(2x) + \tfrac{1}{2}b\sinh(2x)}$$

Method II (harder): Two linearly independent solutions to the equation are $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$. In the general solution $y(x) = \tilde{C}_1 e^{2x} + \tilde{C}_2 e^{-2x}$ we substitute the initial conditions to find the system of linear equations

$$\tilde{C}_1 + \tilde{C}_2 = a \qquad 2\tilde{C}_1 - 2\tilde{C}_2 = b.$$

Solving this pair of linear equations for $\tilde{C}_1$ and $\tilde{C}_2$ gives $\tilde{C}_1 = \frac{2a+b}{4}$, $\tilde{C}_2 = \frac{2a-b}{4}$; the particular solution is

$$\boxed{y(x) = \tfrac{2a+b}{4}e^{2x} + \tfrac{2a-b}{4}e^{-2x}}$$

Notice that

$$a\cosh(2x) + \frac{1}{2}b\sinh(2x) = a\left(\frac{e^{2x}+e^{-2x}}{2}\right) + \frac{b}{2}\left(\frac{e^{2x}-e^{-2x}}{2}\right) = \left(\frac{a}{2}+\frac{b}{4}\right)e^{2x} + \left(\frac{a}{2}-\frac{b}{4}\right)e^{-2x}$$

which shows that the solutions obtained by Methods I and II are actually the same.

2. (a) For an arbitrary real number $c > 3$, find the general solution to the ODE

$$y'' + 2cy' + 9y = 0.$$

Your answer should depend on c.

Answer: The characteristic equation $r^2 + 2cr + 9 = 0$ has roots

$$r = \frac{-2c \pm \sqrt{4c^2 - 36}}{2} = -c \pm \sqrt{c^2 - 9}.$$

The general solution to the equation is

$$\boxed{y(x) = C_1 e^{(-c+\sqrt{c^2-9})x} + C_2 e^{(-c-\sqrt{c^2-9})x}}$$

(b) What happens if you let $c \to 3$ in your answer to part (a)? Find the general solution to the ODE

$$y'' + 6y' + 9y = 0.$$

Answer: When $c \to 3$ the roots of the characteristic equation both tend to $r = -3$. In fact, $r = -3$ is a double root of the characteristic equation $r^2 + 6r + 9 = 0$ for the differential equation $y'' + 6y' + 9y = 0$. The correct general solution to this equation is

$$\boxed{y(x) = C_1 e^{-3x} + C_2 x e^{-3x}}$$