Math 385 Spring 2007
Quiz 5 (Solutions)

1. For a real number x, let $[x]$ be the greatest integer less than or equal to x. For example, $[2.6] = 2$, $[\pi] = 3$, $[4] = 4$, $[-0.7] = -1$, etc.
Let $\{x\} = x - [x]$ be the fractional part of x. For example, $\{2.6\} = 0.6$, $\{\pi\} = \pi - 3 \approx 0.14159\ldots$, $\{4\} = 0$, $\{-0.7\} = 0.3$, etc.

(a) Is $f(x) = [x]$ periodic or not periodic? If it is periodic, state its period.

Answer: f is **not** periodic. f is a nondecreasing function ($x \leq y$ implies $f(x) \leq f(y)$); if f were also periodic it would have to be constant. Since f is obviously not constant (see for example the values above), it cannot be periodic.

(b) Is $g(x) = \{x\}$ periodic or not periodic? If it is periodic, state its period.

Answer: g is periodic, with period $P = 1$. In fact,

$$g(x+1) = \{x+1\} = x + 1 - [x+1] = x + 1 - [x] = 1 = x - [x] = \{x\} = g(x)$$

for all x. Since $g(x) = x$ for all $0 < x < 1$ is strictly increasing on this interval, no value less than one is a period of g.

2. The eigenvalues of

$$y'' + \lambda y = 0, \qquad y(0) = 0, y(5) = 0,$$

are all nonnegative. Determine whether $\lambda = 0$ is an eigenvalue; if so, find an eigenfunction. Next, find all values $\lambda > 0$ which are eigenvalues, and for each such value, find an eigenfunction.

Answer: The general solution to $y'' = 0$ is $y(x) = C_1 x + C_2$; using the boundary conditions gives $0 = y(0) = C_2$ and $0 = y(5) = 5C_1$ so $C_1 = 0$. Thus the only solution in this case is the trivial solution $y(x) \equiv 0$, and $\lambda = 0$ is not an eigenvalue

Next, assume $\lambda > 0$. Write $\lambda = \alpha^2$ with $\alpha > 0$. The general solution to $y'' + \alpha^2 y = 0$ is $y(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$; using the boundary conditions gives $0 = y(0) = C_1(1) + C_2(0) = C_1$ and $0 = y(5) = C_2 \sin(5\alpha)$. The case $C_2 = 0$ gives the trivial solution $y(x) \equiv 0$; we ignore this and concentrate on the case $\sin(5\alpha) = 0$. The latter equation holds only if $5\alpha = n\pi$ for some $n = 1, 2, 3, \ldots$. (Note: we ruled out $n = 0$ since we previously assumed $\alpha > 0$.) The eigenvalues for the problem are

$$\lambda_n = \alpha_n^2 = \frac{n^2\pi^2}{25},$$

and the eigenfunction corresponding to λ_n is

$$y_n(x) = \sin(\alpha_n x) = \sin(\frac{n\pi x}{5}).$$