Study Guide for Exam 2

Prof. Tyson, Math 385 (Spring 2007)

Exam 2 will cover material from IODE Project #2 and sections 2.4, 2.5 and Chapter 3 of Edwards & Penney. In **Chapter 3** (sections 3.1–3.6) we studied higher-order linear equations (forced and unforced), with applications to mechanical oscillations. Section 3.8 on endpoint problems and eigenvalues will **not** be tested on Exam #2.

Topics Covered:

2.4/2.5 (and IODE Project #2) Euler's method and improved Euler's method

3.1/3.2 higher-order linear differential equations (forced & unforced, principle of superposition/linearity, Existence and Uniqueness Theorem, linear independence)

3.3 solving constant-coefficient linear DE's via characteristic equations, differential operators

3.5 forced equations (particular vs. complementary solutions, method of undetermined coefficients, method of variation of parameters)

3.4/3.6 applications (mechanical oscillations, pendulum motion, electrical circuits, etc.), simple harmonic motion, damped vs. undamped motion, resonance

You should be able to:

- run Euler's method to approximate the solution to a simple first-order differential equation, discuss and interpret data related to Euler's method and the improved Euler's method, use the Error Bound Theorems for these methods
- find the general/particular solution to a linear equation (any order), either forced or unforced
- test linear independence of a set of functions
- set up a differential equation to model oscillation, interpret the solution
- distinguish undamped/underdamped/crit. damped /overdamped motion, recognize resonant solutions
- recognize transient vs. steady-state solutions for damped and forced motion, practical resonance

This is not a comprehensive list! However, if you are comfortable with all of these concepts and skills you should be in good shape for the exam.

Use of a calculator will **not** be allowed on the exam.

Practice Problems for Exam #2

1. Explain in your own words why the improved Euler method (see IODE Project #2) is typically more accurate than the original Euler method for approximating solutions to first-order differential equations.

2. Find the general solution to $y'' - 3y' + 2y = e^x$.

3. (a) For which value(s) of ω will resonance occur in the mechanical oscillation equation

$$x'' + 9x = 2\sin(\omega t) - \sin(5\omega t)?$$

(b) Find the solution to the oscillation equation

$$x'' + 9x = 2\sin t - \sin 5t$$

with initial conditions $x(0) = 0$ and $x'(0) = 0$.

4. (a) Show that $y_1(x) = e^{2x}$, $y_2(x) = \cos x$ and $y_3(x) = \sin x$ are linearly independent on the real line.

(b) Write a constant coefficient linear differential equation with general solution $y(x) = C_1y_1(x) + C_2y_2(x) + C_3y_3(x)$, where y_1, y_2, y_3 are the functions in part (a).

(c) Find the solution to the equation in part (b) which satisfies $y(0) = 0$, $y'(0) = -1$ and $y''(0) = 6$.

Answers

1. (Answers may vary) Consider the first-order differential equation $y' = F(x, y)$ with step size h. In Euler's method, the value of the approximate solution at the updated point $x_{n+1} = x_n + h$ is computed using only the value $F(x_n, y_n)$ of the slope field at the original point (x_n, y_n):

$$y_{n+1} = y_n + F(x_n, y_n) \cdot h.$$

In the improved Euler's method, the values of the slope field at both the original point (x_n, y_n) and the update point from Euler's method $(x_n + h, y_n + F(x_n, y_n) \cdot h)$ are averaged to compute a new slope:

$$\frac{1}{2}(k_1 + k_2),$$

where $k_1 = F(x_n, y_n)$ and $k_2 = F(x_n + h, y_n + F(x_n, y_n) \cdot h)$. Then this new slope is used to compute a new update point:

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2) \cdot h.$$

By using a preliminary estimate for the update point to recalculate the slope field, the improved Euler method does a better job of estimating the variation which occurs over the update region $[x_n, x_n + h]$. In practice, the values obtained by the improved Euler method are more accurate approximations to the true solution of the equation.

2. The complementary equation $y_c'' - 3y_c' + 2y_c = 0$ has characteristic equation $r^2 - 3r + 2 = 0$ with roots $r = 1$ and $r = 2$, and hence has solution

$$y_c(x) = C_1 e^x + C_2 e^{2x}.$$

Since the term e^x is duplicated in the complementary solution and in the forcing terms of the original equation $y'' - 3y' + 2y = e^x$, we choose a trial solution $y_p(x) = Axe^x$ for the method of undetermined coefficients. Compute $y_p'(x) = Axe^x + Ae^x$ and $y_p''(x) = Axe^x + 2Ae^x$. Then

$$\begin{aligned} e^x &= y_p''(x) - 3y_p'(x) + 2y_p(x) \\ &= (Axe^x + 2Ae^x) - 3(Axe^x + Ae^x) + 2(Axe^x) = -Ae^x \end{aligned}$$

so $A = -1$ and $y_p(x) = -xe^x$. The general solution to the original equation is $\boxed{y(x) = C_1 e^x + c_2 e^{2x} - xe^x}$.

3. (a) Resonance occurs when one of the forcing terms has a frequency equal to the unforced frequency. From the complementary equation $x_c'' + 9x_c = 0$ we find that the unforced frequency is $\omega_0 = \sqrt{9} = 3$. Resonance occurs when either $\omega = \omega_0$ or $5\omega = \omega_0$, i.e., for $\boxed{\omega = 3}$ or $\boxed{\omega = 3/5}$.

(b) Solve the complementary equation: $x_c(t) = C_1 \cos 3t + C_2 \sin 3t$. Choose a trial solution $x_p(t) = A \sin t + B \sin 5t$ for the method of undetermined coefficients. (It is not necessary to include the corresponding cosine terms in this trial, since the original equation has no damping term.) Then

$$\begin{aligned} x_p''(t) + 9x_p(t) &= (-A \sin t - 25B \sin 5t) + 9(A \sin t + B \sin 5t) \\ &= 8A \sin t - 16B \sin 5t = 2 \sin t - \sin 5t \end{aligned}$$

so $A = \frac{1}{4}$ and $B = \frac{1}{16}$ and $x_p(t) = \frac{1}{4} \sin t + \frac{1}{16} \sin 5t$. The general solution is

$$x(t) = C_1 \cos 3t + C_2 \sin 3t + \frac{1}{4} \sin t + \frac{1}{16} \sin 5t.$$

Imposing the initial conditions gives $0 = x(0) = C_1$ and

$$0 = x'(0) = 3C_2 + \frac{1}{4} + \frac{5}{16} \quad \Rightarrow \quad C_2 = -\frac{3}{16}$$

so $\boxed{x(t) = \frac{1}{4} \sin t - \frac{3}{16} \sin 3t + \frac{1}{16} \sin 5t}$.

4. *Solution:* (a) *Method I.* To show that y_1, y_2, y_3 are linearly independent on the real line, compute the Wronskian

$$\begin{aligned} W(x) &= \begin{vmatrix} e^{2x} & \cos x & \sin x \\ 2e^{2x} & -\sin x & \cos x \\ 4e^{2x} & -\cos x & -\sin x \end{vmatrix} \\ &= e^{2x} \begin{vmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{vmatrix} - 2e^{2x} \begin{vmatrix} \cos x & \sin x \\ -\cos x & -\sin x \end{vmatrix} \\ &\quad + 4e^{2x} \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} \\ &= e^{2x}(\cos^2 x + \sin^2 x) - 2e^{2x}(0) + 4e^{2x}(\cos^2 x + \sin^2 x) \\ &= 5e^{2x} \end{aligned}$$

Since $W(x)$ is not equal to zero for any real number x, y_1, y_2, y_3 are linearly independent on the real line.

Method II. Suppose that $C_1 + C_2 e^x + C_3 xe^x$ is identically equal to zero for some constants C_1, C_2, C_3. Letting $x \to \infty$ and using L'Hopital's rule to compute $\lim_{x \to -\infty} xe^x = 0$, we see that C_1 must equal zero. Thus we must have $C_2 e^x + C_3 xe^x = e^x(C_2 + C_3 x)$ identically equal to zero. Evaluating at $x = 0$ tells us that $C_2 = 0$. Thus $C_3 xe^x$ must be identically equal to zero, which tells us that C_3 is equal to zero. Hence y_1, y_2, y_3 are linearly independent.

(b) The roots of the characteristic equation are $r = 2$, $r = i$ and $r = -i$. This means that the characteristic equation is

$$p(r) = (r-2)(r-i)(r+i) = (r-2)(r^2+1) = r^3 - 2r^2 + r - 2,$$

so the differential equation is $\boxed{y''' - 2y'' + y' - 2y = 0}$.

(c) With $y(x) = C_1 e^{2x} + C_2 \cos x + C_3 \sin x$ we find

$$\begin{aligned} 0 &= y(0) = C_1 + C_2 \\ -1 &= y'(0) = 2C_1 + C_3 \\ 6 &= y''(0) = 4C_1 - C_2. \end{aligned}$$

Solving this system of equations gives $C_1 = \frac{6}{5}$, $C_2 = -\frac{6}{5}$ and $C_3 = -\frac{17}{5}$, so $\boxed{y(x) = \frac{6}{5} e^{2x} - \frac{6}{5} \cos x - \frac{17}{5} \sin x}$.