Homework #10 Solutions

9.1 #14 $\quad f(t) = \begin{cases} 3 & -\pi < t < 0 \\ -2 & 0 < t < \pi \end{cases}$

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}(3\pi - 2\pi) = 1.$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = \frac{3}{\pi}\int_{-\pi}^{0}\cos(nt)\,dt - \frac{2}{\pi}\int_0^{\pi}\cos(nt)\,dt = 0$$

$$B_n = \frac{3}{\pi}\int_{-\pi}^{0}\sin(nt)\,dt - \frac{2}{\pi}\int_0^{\pi}\sin(nt)\,dt = -\frac{3}{\pi}\cdot\frac{1-\cos(n\pi)}{n} + \frac{2}{\pi}\,\frac{\cos(n\pi)-1}{n}$$

$$= -\frac{5}{n\pi}\left(1-\cos(n\pi)\right) = \begin{cases} 0 & n \text{ even} \\ -\frac{10}{n\pi} & n \text{ odd} \end{cases}$$

$$\boxed{f(t) \sim \frac{1}{2} - \frac{10}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin(nt)}$$

#19 $\quad f(t) = \begin{cases} \pi + t & -\pi < t < 0 \\ 0 & 0 < t < \pi \end{cases}$

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+t)\,dt = \frac{1}{\pi}\left(\pi t + \tfrac{1}{2}t^2\right)\Big|_{-\pi}^{0} = \frac{1}{2}\pi$$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+t)\cos(nt)\,dt = \frac{1}{\pi}\left(\frac{(\pi+t)\sin(nt)}{n} + \frac{\cos(nt)}{n^2}\right)\Big|_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left(0 + \frac{1}{n^2} - 0 - \frac{\cos(n\pi)}{n^2}\right) = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n^2\pi} & n \text{ odd} \end{cases}$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{0}(\pi+t)\sin(nt)\,dt = \frac{1}{\pi}\left(-\frac{(\pi+t)\cos(nt)}{n} + \frac{\sin(nt)}{n^2}\right)\Big|_{-\pi}^{0}$$

$$= \frac{1}{\pi}\left(-\frac{\pi}{n} + 0 + 0 - 0\right) = -\frac{1}{n}.$$

$$\boxed{f(t) \sim \frac{\pi}{4} + \frac{2}{\pi}\sum_{n \text{ odd}} \frac{1}{n^2}\cos(nt) - \sum_{n=1}^{\infty}\frac{1}{n}\sin(nt)}$$

#25 $f(t) = \cos^2 2t$.

Apply a trig identity : $\cos^2 2t = \frac{1}{2} + \frac{1}{2}\cos(4t)$

Fourier series $\boxed{f(t) = \frac{1}{2} + \frac{1}{2}\cos(4t)}$

9.2 #1 $f(t) = \begin{cases} -2 & -3<t<0 \\ 2 & 0<t<3 \end{cases}$ odd function: $A_n = 0 \ \forall n$

$B_n = \frac{1}{3}\int_{-3}^{3} f(t)\sin\left(\frac{n\pi t}{3}\right)dt = \frac{2}{3}\int_0^3 f(t)\sin\left(\frac{n\pi t}{3}\right)dt$

$= \frac{4}{3}\int_0^3 \sin\left(\frac{n\pi t}{3}\right)dt = \left(\frac{4}{3}\right)\left(\frac{3}{n\pi}\right)\left(-\cos\left(\frac{n\pi t}{3}\right)\right)\Big|_0^3$

$= -\frac{4}{n\pi}(\cos(n\pi) - 1) = \begin{cases} 0 & n \text{ even} \\ +\frac{8}{n\pi} & n \text{ odd} \end{cases}$

$$\boxed{f(t) \sim +\frac{8}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin\left(\frac{n\pi t}{3}\right)}$$

#6 $f(t) = t \quad 0<t<3$ period $P = 3$, $L = \frac{3}{2}$!

$A_0 = \frac{2}{3}\int_0^3 f(t)\,dt = \frac{2}{3}\int_0^3 t\,dt = \frac{2}{3}\cdot\frac{9}{2} = 3$

$A_n = \frac{2}{3}\int_0^3 t\cos\left(\frac{2n\pi t}{3}\right)dt = \left(\frac{2}{3}\right)\left(\frac{3}{2\pi n}\right)^2\int_0^{2n\pi} u\cos u\,du \quad \left(u = \frac{2n\pi t}{3}\right)$

$= \frac{3}{2\pi^2 n^2}(u\sin u + \cos u)\Big|_0^{2n\pi} = 0$

$B_n = \left(\frac{2}{3}\right)\left(\frac{3}{2\pi n}\right)^2\int_0^{2\pi} u\sin u\,du = \frac{3}{2\pi^2 n^2}(-u\cos u + \sin u)\Big|_0^{2n\pi}$

$= \frac{3}{2\pi^2 n^2}(-2n\pi) = -\frac{3}{\pi n}$ $\boxed{f(t) \sim \frac{3}{2} - \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi t}{3}\right)}$