Homework # 11 Solutions

9.3 #14 $x'' + 2x = t \qquad x(0) = x(2) = 0$

Fourier Sine series for $f(t) = t \quad 0 < t < 2$:

$$f(t) \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi t}{2}\right) \qquad (\text{see Ex } 1, \S 9.3)$$

Guess $x_p(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{2}\right)$

$$\left(-\frac{n^2\pi^2}{4} + 2\right) B_n = \frac{4(-1)^{n+1}}{\pi n} \qquad \text{for all } n = 1, 2, 3, \dots$$

$$B_n = \frac{16(-1)^{n+1}}{\pi n (8 - n^2\pi^2)}$$

$$\boxed{x(t) = \frac{16}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(8 - n^2\pi^2)} \sin\left(\frac{n\pi t}{2}\right)}$$

#15 $x'' + 2x = t \qquad x'(0) = x'(\pi) = 0$

Fourier cosine series for $f(t) = t \quad 0 < t < \pi$:

$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \cos(nt)$$

Guess $x_p(t) = \frac{A_0}{2} + \sum_{n \text{ odd}} A_n \cos(nt)$

$$(-n^2 + 2) A_n = -\frac{4}{n^2\pi} \qquad \text{for all } n = 1, 3, 5, 7, \dots$$

and $A_0 = \frac{\pi}{2}$

$$A_n = \frac{4}{\pi n^2 (n^2 - 2)} \qquad n = 1, 3, 5, 7, \dots$$

$$\boxed{x(t) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2(n^2-2)} \cos(nt)}$$

#18 Termwise differentiation gives

$$-\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(n\pi t) - 4 \sum_{n=1}^{\infty} \cos(n\pi t)$$

This series does not converge at some values of t (e.g. $t=0$).

9.4 #7 Natural frequency is $\omega_0 = 3$

$F(t) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin(nt)$ Fourier sine series

A nontrivial $\sin(3t)$ occurs, so $\boxed{\text{resonance occurs}}$

#8 Natural frequency is $\omega_0 = \sqrt{5}$

$F(t)$ = some Fourier series involving terms $\sin(n\pi t)$, $n=1,3,5,\ldots$

Since $n\pi \neq \sqrt{5}$ for any integer n,

$\boxed{\text{resonance does not occur}}$

9.5 #10 $u_t = \frac{1}{5} u_{xx}$ $0 < x < 10, t > 0$ $u(0,t) = u(10,t) = 0$ $u(x,0) = 4x$

Fourier sine series $4x = u(x,0) \sim \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{10}\right)$

Solution:

$$u(x,t) = \frac{80}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{n^2\pi^2 t}{500}} \sin\left(\frac{n\pi x}{10}\right)$$

#11 $u_t = \frac{1}{5} u_{xx}$ $\quad 0<x<10, \; t>0 \quad u_x(0,t) = u_x(10,t) = 0 \quad u(x,0) = 4x$

Fourier cosine series $\quad 4x = u(x,0) \sim 20 - \frac{160}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos\left(\frac{n\pi x}{10}\right)$

Solution:

$$u(x,t) = 20 - \frac{160}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} e^{-\frac{n^2\pi^2 t}{500}} \cos\left(\frac{n\pi x}{10}\right)$$

#14 $L = 50\text{cm}$ $\quad f(x) = 2x \,^\circ C \quad u_x(0,t) = u_x(50,t) = 0$

copper $k = 1.15 \text{ cm}^2/\text{sec}$

(a) Fourier cosine series $\quad 2x = u(x,0) \sim 50 - \frac{400}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} \cos\left(\frac{n\pi x}{50}\right)$

$$u(x,t) = 50 - \frac{400}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} e^{-\frac{n^2\pi^2 k t}{2500}} \cos\left(\frac{n\pi x}{50}\right)$$

(b) $$u(10,60) = 50 - \frac{400}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} e^{-\frac{69 n^2\pi^2}{2500}} \cos\left(\frac{n\pi}{5}\right)$$

$$\approx 50 - 24.9698 + 0.1199 + 0.0018 + \cdots$$

$$\approx 25.15^\circ C$$

(c) How long until $u(10,t) = 45^\circ C$?

Use first two terms

$$50 - \frac{400}{\pi^2} e^{-\frac{1.15\pi^2 t}{2500}} \cos\left(\frac{\pi}{5}\right) = 45$$

Solve for t: $t \approx 414.23$ seconds $= \boxed{6 \text{ min } 54 \text{ sec}}$

Check: Keep 3 terms

$$50 - \frac{400}{\pi^2} e^{-\frac{1.15\pi^2 t}{2500}} \cos\left(\frac{\pi}{5}\right) - \frac{400}{9\pi^2} e^{-\frac{9(1.15)\pi^2 t}{2500}} \cos\left(\frac{3\pi}{5}\right) = 45$$

Solve (with a graphing calculator):

$t \approx 414.23$ seconds.