Homework #9 Solutions

3.8 #2 $y''+\lambda y=0 \qquad y'(0)=0 \qquad y'(\pi)=0$

Try $\lambda=0$. $\qquad y''=0 \Rightarrow y(x)=C_1+C_2x$

$$y'(x)=C_2$$

$0=y'(0)=C_2$ also $0=y'(\pi)=C_2$

We only get $C_2=0$. No restriction on C_1.

$y(x)=C_1$ is a solution for any value of C_1

In particular, we can choose $C_1\neq 0$ so $\boxed{\lambda=0 \text{ is an eigenvalue}}$

Try $\lambda>0$, $\lambda=\alpha^2$, $\alpha\neq 0$.

$$y''+\alpha^2 y=0 \Rightarrow y(x)=C_1\cos(\alpha x)+C_2\sin(\alpha x)$$

$$y'(x)=-\alpha C_1\sin(\alpha x)+\alpha C_2\cos(\alpha x)$$

$0=y'(0)=\alpha C_2 \Rightarrow C_2=0$

$0=y'(\pi)=-\alpha C_1\sin(\alpha\pi) \Rightarrow$ either $C_1=0$ or $\sin(\alpha\pi)=0$

$C_1=0$ trivial solution $\rightarrow$ disregard

$\sin(\alpha\pi)=0 \Rightarrow \alpha\in\{\ldots,-3,-2,-1,1,2,3,\ldots\}$ (remember $\alpha\neq 0$)

$\lambda = 1, 4, 9, 16, \ldots$ are eigenvalues

$\lambda_n = n^2$, $n = 1, 2, 3 \ldots$ with associated eigenfunction

$y_n(x) = C_1 \cos(nx)$, $C_1 \neq 0$.

\#6 $y'' + \lambda y = 0 \qquad y'(0) = 0 \qquad y(1) + y'(1) = 0$

a) Try $\lambda = 0 \qquad y'' = 0 \Rightarrow y(x) = C_1 + C_2 x$

$y'(x) = C_2$

$0 = y'(0) = C_2$
$0 = y(1) + y'(1) = C_1$ $\Rightarrow$ trivial solution

$\lambda = 0$ is not an eigenvalue.

b) Try $\lambda = \alpha^2 > 0 \qquad y(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$

$y'(x) = -\alpha C_1 \sin(\alpha x) + \alpha C_2 \cos(\alpha x)$

$0 = y'(0) = \alpha C_2 \Rightarrow C_2 = 0$

$0 = y(1) + y'(1) = C_1 \cos(\alpha) - \alpha C_1 \sin(\alpha)$

$\Rightarrow C_1 = 0$ or $\cos(\alpha) = \alpha \sin(\alpha)$

$C_1 = 0$ trivial solution $\rightarrow$ disregard

$\cos(\alpha) = \alpha \sin(\alpha) \iff \tan(\alpha) = \frac{1}{\alpha}$.

Find

$\alpha > 0$ s.t. $\tan(\alpha) = \frac{1}{\alpha}$.

Figure 3.8.9 on p. 240 indicates that there are positive solutions

$$0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots$$

to this equation. (Values of α_n cannot be determined explicitly.)

For such α_n, $\lambda_n = \alpha_n^2$ is an eigenvalue, with eigenfunction

$$y_n(x) = c_1 \cos(\alpha_n x).$$

#13 $y'' + 2y' + \lambda y = 0$ $\qquad y(0) = y(1) = 0$

a) Try $\lambda = 1$. $\quad y'' + 2y' + y = 0$

$$y(x) = C_1 e^{-x} + C_2 x e^{-x}$$

$$0 = y(0) = C_1$$
$$0 = y(1) = C_2 e^{-1} \Rightarrow C_2 = 0 \Rightarrow \text{trivial solution}$$

b) Try $\lambda < 1$. Set $\lambda = 1 - \alpha^2$, $\alpha \neq 0$.

$$y'' + 2y' + (1 - \alpha^2) y = 0$$

characteristic equation $r^2 + 2r + 1 - \alpha^2 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4(1-\alpha^2)}}{2} = -1 \pm \alpha.$$

$$y(x) = C_1 e^{(-1+\alpha)x} + C_2 e^{(-1-\alpha)x}$$

$$\left.\begin{aligned} 0 &= y(0) = C_1 + C_2 \\ 0 &= y(1) = (\alpha-1)\, C_1 - (\alpha+1)\, C_2 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= 0 \\ C_2 &= 0 \end{aligned}$$

trivial solution

c) Try $\lambda > 1$. Set $\lambda = 1 + \alpha^2$, $\alpha \neq 0$.

$$y'' + 2y' + (1+\alpha^2)y = 0$$

characteristic equation $r^2 + 2r + 1 + \alpha^2 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4(1+\alpha^2)}}{2} = -1 \pm \alpha i$$

$$y(x) = C_1 e^{-x} \cos(\alpha x) + C_2 e^{-x} \sin(\alpha x)$$

$$0 = y(0) = C_1$$
$$0 = y(1) = C_2 e^{-1} \sin(\alpha) \Rightarrow \underset{\text{(disregard)}}{C_1 = 0} \text{ or } \sin(\alpha) = 0$$

$\sin(\alpha) = 0 \Rightarrow \alpha \in \{\ldots, -3\pi, -2\pi, -\pi, \pi, 2\pi, 3\pi, \ldots\}$ (recall $\alpha \neq 0$)

$$\lambda = 1+\alpha^2 \in \{ 1+\pi^2, 1+4\pi^2, 1+9\pi^2, 1+16\pi^2, \dots \}$$

$\lambda_n = 1 + n^2\pi^2 \quad n = 1, 2, 3, \dots$ eigenvalues

with eigenfunctions

$$y_n(x) = C_2 e^{-x} \sin(n\pi x).$$

9.1 # 27 Prove:

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \pi \cdot \delta_{mn} \qquad \delta_{mn} = \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$$

Proof: $\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos((m+n)t) + \cos((m-n)t)\right] dt$

If $m \neq n$, this is $= \frac{1}{2}\left[\frac{\sin((m+n)t)}{m+n} + \frac{\sin((m-n)t)}{m-n}\right]_{-\pi}^{\pi}$

$$= \frac{1}{2}\left[\frac{\sin(m\pi+n\pi)}{m+n} + \frac{\sin(m\pi-n\pi)}{m-n} - \frac{\sin(-m\pi-n\pi)}{m+n} - \frac{\sin(-m\pi+n\pi)}{m-n}\right]$$

$$= 0.$$

If $m=n$, this is $= \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(2nt)+1\right] dt$

$$= \frac{1}{2}\left[\frac{\sin(2nt)}{2n} + t\right]_{-\pi}^{\pi} = \frac{1}{2}\left[\frac{\sin(2n\pi)}{2n} + \pi - \frac{\sin(-2n\pi)}{2n} - (-\pi)\right]$$

$$= \frac{1}{2}(2\pi) = \pi.$$

Math 385 Spring 2007
Extra problem for Homework #9

A positive function w defined on an interval $[a, b]$ is called a *weight.* We say that two functions f and g are *orthogonal on the interval* $[a, b]$ *with respect to the weight* $w(x)$ if

$$\int_a^b f(x)g(x)w(x)\,dx = 0.$$

(a) Show that the functions $f(x) = x$ and $g(x) = 2x^2 - 1$ are orthogonal on the interval $[-1, 1]$ with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$. (Hint: make the substitution $x = \cos\theta$ in the integral.)

(b) Repeat part (a) for the functions $f(x) = x$ and $h(x) = 4x^3 - 3x$. (Hint: the same substitution is still helpful.)

(c) Show that $g(\cos\theta) = \cos(2\theta)$ and $h(\cos\theta) = \cos(3\theta)$ for every θ.

(d) (EXTRA CREDIT) For each positive integer n, there is a polynomial T_n of degree n satisfying $T_n(\cos\theta) = \cos(n\theta)$ for all θ. Show that T_n and T_m are orthogonal on $[-1, 1]$ with respect to the weight $w(x) = \frac{1}{\sqrt{1-x^2}}$ if $m \neq n$. Calculate

$$\int_{-1}^{1} T_n(x)^2 w(x)\,dx.$$

Part (c) shows that $T_2(x) = 2x^2 - 1$ and $T_3(x) = 4x^3 - 3x$. Calculate $T_4(x)$.

Remark. T_n is called the *nth Chebyshev polynomial of the first kind.* Chebyshev polynomials are used to find best approximating polynomials for non-polynomial functions. See

`mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html`

and

`mathworld.wolfram.com/ChebyshevApproximationFormula.html`

for more information.

Extra Problem

a) $\int_{-1}^{1} x(2x^2-1)\frac{1}{\sqrt{1-x^2}}\,dx$ $\qquad x = \cos\theta$, $dx = -\sin\theta\,d\theta$

$= \int_{\pi}^{0} (\cos\theta)(2\cos^2\theta-1)\left(\frac{1}{\sin\theta}\right)(-\sin\theta\,d\theta)$

when $x=-1$, $\theta=\pi$
when $x=1$, $\theta=0$

$= \int_0^{\pi} \cos\theta\cos(2\theta)\,d\theta$

$= \frac{1}{2}\int_0^{\pi}[\cos 3\theta + \underset{\cos\theta}{\underset{\parallel}{\cos(-\theta)}}]\,d\theta$ $\qquad$ using the same trig identity $\cos A\cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)]$

$= \frac{1}{2}\left[\frac{\sin(3\theta)}{3} + \sin\theta\right]_0^{\pi}$

$= \frac{1}{2}\left[\frac{\sin(3\pi)}{3} + \sin\pi\right] = 0.$

b) $\int_{-1}^{1} x(4x^3-3x)\frac{1}{\sqrt{1-x^2}}\,dx$

$= \int_{\pi}^{0} (\cos\theta)(4\cos^3\theta-3\cos\theta)\left(\frac{1}{\sin\theta}\right)(-\sin\theta\,d\theta)$

$= \int_0^{\pi} \cos\theta\cos(3\theta)\,d\theta = --- = 0$ (same idea as above)

c) $g(\cos\theta) = 2\cos^2\theta - 1 = \cos(2\theta)$ $\qquad$ double angle formula for cos

$h(\cos\theta) = 4\cos^3\theta - 3\cos\theta = \cos(3\theta)$ $\qquad$ triple angle formula for cos.