Math 416 HW #5
due Friday, 3/7

1: Friedberg–Insel–Spence, §2.6 #13 (a)(b)(d)(e)

2: (a) Let O_n denote the $n \times n$ matrix all of whose entries are zero. A matrix $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ is called **nilpotent** if $\mathbf{A}^k = O_n$ for some integer $k \geq 1$. Prove that if $\mathbf{A}$ is nilpotent, then $\det \mathbf{A} = 0$.

(b) A matrix $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ is called **orthogonal** if $\mathbf{A}^T\mathbf{A} = I_n$. Prove that if $\mathbf{A}$ is orthogonal, then $|\det \mathbf{A}| = 1$.

(c) A matrix $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ is called **skew-symmetric** if $\mathbf{A}^T = -\mathbf{A}$. Prove that if $\mathbf{A}$ is skew-symmetric and n is odd, then $\det \mathbf{A} = 0$.

(d) Give an example of a skew-symmetric matrix $\mathbf{A} \in M_{2\times 2}(\mathbb{R})$ with $\det \mathbf{A} = 1$.

3: (a) Prove that

$$\det \begin{pmatrix} a & b \\ b & a \end{pmatrix} = a^2 - b^2$$

and

$$\det \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} = a^3 - 3ab^2 + 2b^3 = (a-b)^2(a+2b).$$

(b) Let $\mathbf{A}$ be the $n \times n$ matrix which has diagonal entries equal to a and all other entries equal to b. Prove that $\det \mathbf{A} = (a-b)^{n-1}(a+(n-1)b)$.

Hint: Let $F_n(a,b)$ be the determinant of this matrix $\mathbf{A}$. Show that

$$F_n(a,b) = \left(1 - \frac{b}{a}\right)^{n-1} F_{n-1}(a+b, b)$$

whenever $a \neq 0$. Then prove the desired result in case $a \neq 0$ by induction. Finally, argue by continuity that the desired result must hold also in case $a = 0$.

4: (a) Let $A, B, C, D \in M_{2\times 2}(\mathbb{R})$. Show by an example that the identity

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det D - \det B \det C$$

can fail to hold.

(b) Prove that

$$\det \begin{pmatrix} A & B \\ O_2 & D \end{pmatrix} = \det A \det D$$

whenever $A, B, D \in M_{2\times 2}(\mathbb{R})$.