Math 441 Fall 2005
First Midterm Exam Solutions

Problem 1. Solve the initial value problem

$$y' - 4y = e^{5x}, \qquad y(0) = 3.$$

Answer: The integrating factor is $I(x) = e^{-\int 4\,dx} = e^{-4x}$. Multiplying through by $I(x)$ gives

$$(e^{-4x}y)' = e^{-4x}y' - 4e^{-4x}y = e^{5x}e^{-4x} = e^{x}$$

so

$$e^{-4x}y = e^{x} + C.$$

The initial value gives $3 = 1 + C$ so $C = 2$ and

$$y(x) = e^{5x} + 2e^{4x}.$$

Problem 2. Consider the differential equation

$$y + (2x - \frac{1}{y} + 2)\,\frac{dy}{dx} = 0.$$

(a) Show that this equation is not exact.

Answer: Set $M(x,y) = y$ and $N(x,y) = 2x - \frac{1}{y} + 2$. Since $M_y = 1 \neq 2 = N_x$, the equation is not exact.

(b) Show that this equation becomes exact when multiplied by a suitable integrating factor $I(y)$ depending only on y, and solve the resulting equation.

Answer: Let $I(y) = y$ and consider

$$y^2 + (2xy - 1 + 2y)\,\frac{dy}{dx} = 0.$$

Set $M(x,y) = y^2$ and $N(x,y) = 2xy - 1 + 2y$. Since $M_y = 2y = N_x$, the equation is now exact. The solution is $\psi(x,y) = C$, where $\psi_x = M$ and $\psi_y = N$. We compute

$$\psi(x,y) = xy^2 - y + y^2,$$

so the solution is

$$xy^2 - y + y^2 = C.$$

(c) Solve the initial value problem $y + (2x - \frac{1}{y} + 2)\frac{dy}{dx} = 0$, $y(0) = -1$. At which points (x, y) does the implicit solution *fail* to be the graph of an explicit function $y = f(x)$ locally?

Answer: The initial value gives $C = 2$, so the solution is given implicitly as

$$xy^2 - y + y^2 = 2.$$

Let $F(x, y) = xy^2 - y + y^2 - 2$. This function fails to be a local graph over the x-axis at all points (x, y) where $F(x, y) = 0$ and $\frac{\partial F}{\partial y}(x, y) = 0$, i.e., at points (x, y) satisfying

$$xy^2 - y + y^2 - 2 = 0 \quad \text{and} \quad 2xy - 1 + 2y = 0.$$

We must solve these two equations simultaneously. From the second equation we find

$$x = \frac{1}{2y} - 1;$$

substituting this in the first equation gives

$$\frac{1}{2}y - y - 2 = 0$$

so $y = -4$ and $x = -\frac{9}{8}$. The only point where this solution fails to be a local graph over the x-axis is $(-\frac{9}{8}, -4)$.

Problem 3. Let a, b, c be real numbers satisfying $b^2 > 4ac$. Find all real numbers r so that $y(x) = e^{rx}$ is a solution to the equation

$$ay'' + by' + cy = 0.$$

Answer: Substituting $y(x) = e^{rx}$ in the equation gives

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = (ar^2 + br + c)e^{rx} = 0.$$

Since $e^{rx} \neq 0$ for any x, we must have

$$ar^2 + br + c = 0.$$

Solutions are given by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Problem 4. A cup of hot coffee (with initial temperature 150° F) is left to cool. Room temperature is 75° F. Assume that the coffee cools at a rate proportional to the difference between the current temperature of the coffee and the ambient room temperature.

(a) Write an initial value problem modelling this scenario. (Your answer may involve one or more undetermined parameters.) *Note: do* **not** *give the solution to the differential equation in this part!*

Answer: Let $T(t)$ be the temperature of the coffee at time t. The equation is

$$T' = -k(T - 75), \qquad T(0) = 150,$$

where k is a positive constant.

(b) Solve the differential equation which you found in part (a).

Answer: Separate variables:

$$\frac{dT}{T - 75} = -k\,dt$$

so

$$\ln|T - 75| = -kt + C.$$

The initial condition gives $C = \ln(75)$, so

$$T - 75 = 75e^{-kt}$$

or

$$T(t) = 75 + 75e^{-kt}.$$

Problem 5. (30 points) (a) State the Existence and Uniqueness Theorem for first-order ordinary differential equations.

Answer: Let $F(x, y)$ be defined on a rectangle $R = I \times J = [a, b] \times [c, d]$ in the xy-plane. Assume that F and $\frac{\partial F}{\partial y}$ are continuous on R, and let $(x_0, y_0) \in R$. Then there exists a unique solution $y = f(x)$ to the initial value problem

$$y' = F(x, y), \qquad y(x_0) = y_0$$

defined for $x \in I'$, where I' is some interval contained in I and containing x_0.

In the rest of this problem, consider the differential equation

$$\frac{dy}{dx} = \frac{1}{y}, \qquad y(0) = 1.$$

(b) Solve this initial value problem.

Answer: Separate variables:

$$y\,dy = dx$$

so

$$\frac{1}{2}y^2 = x + C.$$

The initial condition gives $C = \frac{1}{2}$ so

$$y^2 = 2x + 1$$

or

$$y(x) = \sqrt{2x+1}.$$

(We choose the positive square root because of the initial condition $y(0) = +1$.)

(c) On which of the following rectangles $R = I \times J$ are the hypotheses of the Existence and Uniqueness Theorem satisfied for this differential equation? Justify your answer.

(i) $R = I \times J = [-1, 3] \times [-2, 4]$
(ii) $R = I \times J = [-2, 2] \times [\frac{1}{2}, 5]$
(iii) $R = I \times J = [1, 3] \times [1, 4]$

Answer: (i) is ruled out, since F is not continuous along the line $y = 0$. (iii) is ruled out, since $(x_0, y_0) = (0, 1)$ is not in R. The correct answer is (ii); F and $\frac{\partial F}{\partial y}$ are continuous on R and $(0, 1) \in R$.

(d) Let I be the interval along the x-axis which you found in part (c). What is the largest subinterval $I' \subset I$ containing $x_0 = 0$ on which the solution to the equation is defined? Why is your answer to this question consistent with the Existence and Uniqueness Theorem?

Answer: The solution $y(x) = \sqrt{1 + 2x}$ is defined for $-\frac{1}{2} \le x \le 2$. The interval of definition for the solution is $I' = [-\frac{1}{2}, 2]$ which is a subinterval of $I = [-2, 2]$ and contains $x_0 = 0$. This agrees with the conclusion of the theorem.