Math 441 Fall 2005
Second Midterm Exam (Solutions)

Problem 1. (25 points) (a) Prove that $y_1(x) = e^{-x}$, $y_2(x) = \cos(x)$, and $y_3(x) = \sin(x)$ are linearly independent on the real line.

Answer: The Wronskian of y_1, y_2 and y_3 is

$$\begin{aligned} W(x) &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-x} & \cos(x) & \sin(x) \\ -e^{-x} & -\sin(x) & \cos(x) \\ e^{-x} & -\cos(x) & -\sin(x) \end{vmatrix} \\ &= e^{-x}(\sin^2 x + \cos^2 x) - 0 + e^{-x}(\cos^2 x + \sin^2 x) \\ &= 2e^{x}. \end{aligned}$$

Since $W(x) \neq 0$ for any real number x, the functions y_1, y_2, y_3 are linearly independent on the entire real line.

(b) Write a constant coefficient linear differential equation whose general solution is the general linear combination of the functions y_1, y_2, y_3 from part (a).

Answer: The roots of the characteristic equation are $r = -1, i, -i$. The characteristic equation is $(r+1)(r^2+1) = 0$, or $r^3 + r^2 + r + 1 = 0$, so the differential equation is $\boxed{y''' + y'' + y' + y = 0}$.

(c) Find the particular solution to your equation from part (b) satisfying the initial conditions $y(0) = 4$, $y'(0) = -5$, $y''(0) = 0$.

Answer: Substitute the initial conditions into $y(x) = C_1 e^{-x} + C_2 \cos x + C_3 \sin x$ to obtain the system of linear equations

$$C_1 + C_2 = 4, \quad -C_1 + C_3 = -5, \quad C_1 - C_2 = 0.$$

The solution to this system is $C_1 = 2$, $C_2 = 2$, $C_3 = -3$, so the particular solution is

$$\boxed{y(x) = 2e^{-x} + 2\cos x - 3\sin x}.$$

Problem 2. (20 points) Find the general solution to the differential equation $y'' - 3y' + 2y = e^x$ by two methods: (a) the method of undetermined coefficients, (b) the method of variation of parameters. Reconcile your answers.

Answer: The complementary equation $y_c'' - 3y_c' + 2y_c = 0$ has characteristic equation $r^2 - 3r + 2 = 0$ with roots $r = 1$ and $r = 2$, and hence has solution $y_c(x) = C_1 e^x + C_2 e^{2x}$.

Method I: Undetermined Coefficients. Since the term e^x is duplicated in the complementary solution and in the forcing terms of the original equation $y'' - 3y' + 2y = e^x$, we choose a trial solution $y_p(x) = Axe^x$. Compute $y_p'(x) = Axe^x + Ae^x$ and $y_p''(x) = Axe^x + 2Ae^x$. Then

$$e^x = y_p''(x) - 3y_p'(x) + 2y_p(x) = (Axe^x + 2Ae^x) - 3(Axe^x + Ae^x) + 2(Axe^x) = -Ae^x$$

so $A = -1$ and $y_p(x) = -xe^x$. The general solution to the original equation is

$$\boxed{y(x) = C_1 e^x + C_2 e^{2x} - xe^x}.$$

Method II: Variation of Parameters. We guess a solution of the form

$$y_p(x) = u_1(x)e^x + u_2(x)e^{2x}.$$

We impose the requirement

$$u_1' e^x + u_2' e^{2x} = 0 \tag{1}$$

which gives

$$y_p'(x) = u_1(x)e^x + 2u_2(x)e^{2x}.$$

Then

$$y_p''(x) = [u_1(x)e^x + 4u_2(x)e^{2x}] + [u_1'(x)e^x + 2u_2'(x)e^{2x}]$$

so

$$y_p'' - 3y_p' + 2y_p = u_1' e^x + 2u_2' e^{2x} = e^x. \tag{2}$$

Solving (1) and (2) simultaneously for u_1' and u_2' gives

$$u_1' = -1 \quad \text{and} \quad u_2' = e^{-x}.$$

Thus

$$u_1(x) = -x \quad \text{and} \quad u_2(x) = -e^{-x},$$

so $y_p(x) = -xe^{-x} - e^{-x}(e^{2x}) = -xe^{-x} - e^x$ and the general solution to the original equation is

$$\boxed{y(x) = C_1 e^x + (C_2 - 1)e^{2x} - xe^x}.$$

This differs from the previous method only in the choice of the arbitrary constants C_1, C_2.

Problem 3. (15 points) (a) For which value(s) of ω will resonance occur in the mechanical oscillation equation

$$x'' + 9x = 2\sin(5\omega t)?$$

Justify your answer.

Solution: Resonance occurs when the forcing term has a frequency equal to the unforced frequency. From the complementary equation $x_c'' + 9x_c = 0$ we find that the unforced frequency is $\omega_0 = \sqrt{9} = 3$. Resonance occurs when $5\omega = \omega_0$, i.e., for $\boxed{\omega = 3/5}$.

(b) A mass on a spring oscillates according to the equation

$$x'' + 9x = 2\sin 5t.$$

It is released from a resting position at the equilibrium location. Find a formula for the position $x(t)$ of the mass at time t.

Solution: The complementary equation (see part (a)) has general solution $x_c(t) = C_1\cos 3t + C_2\sin 3t$. We choose a trial solution $x_p(t) = A\sin 5t$ for the method of undetermined coefficients. (Remember that it is not necessary to include the corresponding cosine terms in this trial solution, since the equation has no damping term.) Then

$$2\sin 5t = x_p''(t) + 9x_p(t) = -25A\sin 5t + 9A\sin 5t = -16A\sin 5t$$

so $A = -\frac{1}{8}$ and $x_p(t) = -\frac{1}{8}\sin 5t$. The general solution is

$$x(t) = C_1\cos 3t + C_2\sin 3t - \frac{1}{8}\sin 5t.$$

Imposing the initial conditions gives

$$0 = x(0) = C_1$$

and

$$0 = x'(0) = 3C_2 - \frac{5}{8} \quad \Rightarrow \quad C_2 = \frac{5}{24}$$

so

$$\boxed{x(t) = \tfrac{5}{24}\sin t - \tfrac{1}{8}\sin 5t}.$$

Problem 4. (15 points) A population of fish in a pond grows according to a logistic model. The carrying capacity of the pond is 1000 fish. At time $t = 0$ the population is 100 fish and is growing at a rate of 20 fish per year. Write an autonomous differential equation (with initial condition) modelling this scenario. Find the critical points for your equation and discuss the stability/instability of each critical point. Justify your answers. Give a rough sketch of the phaseline and vector field portrait for the equation. You **do not** need to solve the equation.

Answer: The logistic model for the growth rate of a population $P(t)$ of individuals with respect to time t is

$$P' = kP(1 - P/M),$$

where k is a proportionality constant and M is the carrying capacity. From the initial conditions $P(0) = 100$ and $P'(0) = 20$ we find

$$20 = P'(0) = k100(1 - 100/1000) = 90k \quad \Rightarrow \quad k = \frac{2}{9}.$$

We obtain the initial value problem

$$P' = \frac{2}{9}P(1 - \frac{P}{1000}), \qquad P(0) = 100.$$

Let $f(P) = \frac{1}{4500}P(1000 - P)$. The critical points are located at the roots of f: $P = 0$ and $P = 1000$. To analyze the stability of these critical points, we compute

$$f(P) = \frac{2}{9}P - \frac{1}{4500}P^2$$

so

$$f'(P) = \frac{2}{9} - \frac{1}{2250}P.$$

Then $f'(0) = \frac{2}{9} > 0$ so the critical point at $P = 0$ is unstable. Also $f'(1000) = -\frac{2}{9} < 0$ so the critical point at $P = 1000$ is stable. The vector field portrait, plaseline and a graph of $f(P)$ vs. P can be found in Figures 2.5.2, 2.5.3 and 2.5.4 in the text (with different notation).

Problem 5. (25 points) (a) Suppose that y_1 and y_2 are two solutions to a second-order linear ODE

$$y'' + p_1(x)y' + p_0(x)y = 0,$$

where p_0 and p_1 are continuous functions defined on an interval I. What can you say about the Wronskian of y_1 and y_2? Under what conditions do y_1 and y_2 form a fundamental set of solutions, and what does this imply about solutions to IVP's of the form

$$y'' + p_1(x)y' + p_0(x)y = 0, \qquad y(x_0) = y_0,\ y'(x_0) = y_0'?$$

Answer: The Wronskian $W(y_1, y_2) = e^{-\int p_1}$ is either identically equal to zero, or never equal to zero throughout the interval I. If $W(y_1, y_2) \neq 0$ in I, then y_1 and y_2 are linearly independent on I and form a fundamental set of solutions. In this case, every solution to

$$y'' + p_1 y' + p_0 y = 0$$

takes the form $y = C_1 y_1 + C_2 y_2$, and every initial value problem

$$y'' + p_1 y' + p_0 y = 0, \qquad y(x_0) = y_0,\ y'(x_0) = y_0',$$

has a unique solution in this form.

In the remainder of this problem, we consider the second-order ODE

$$3x^2 y'' - 4xy' + 4y = 0. \tag{3}$$

(b) What is the largest interval containing the point $x_0 = 1$ on which a pair of linearly independent continuous solutions to (3) is necessarily defined? Justify your answer.

Answer: Rewrite the equation in the form

$$y'' - \frac{4}{3x} y' + \frac{4}{3x^2} y = 0.$$

The coefficients are not continuous at $x = 0$. Thus the largest interval containing $x_0 = 1$ on which a pair of linearly independent continuous solutions is necessarily defined is $(0, \infty)$.

(c) Verify that $y_1(x) = x$ is a solution to (3). Use reduction of order to find a second linearly independent solution y_2 and write the general solution to (3).

Answer: $3x^2 y_1'' - 4xy_1' + 4y_1 = 0 - 4x + 4x = 0$. We guess another solution in the form

$$y_2(x) = u(x) y_1(x) = x u(x).$$

Then $y_2' = xu' + u$ and $y_2'' = xu'' + 2u'$ so

$$0 = 3x^2 y_2'' - 4xy_2' + 4y_2 = 3x^3 u'' + 2x^2 u'.$$

This is a separable equation for u':

$$\frac{u''}{u'} = -\frac{2}{3x},$$

whose solution is $u' = x^{-2/3}$. Thus $u(x) = 3x^{1/3}$. We can drop the coefficient 3 and just take $u(x) = x^{1/3}$. A second solution is $y_2(x) = x^{4/3}$ and the general solution is

$$y(x) = C_1 x + C_2 x^{4/3}.$$