

## Math 441 Fall 2005 Quiz #3 Solutions

1. Two tanks (Tank A and Tank B) containing salt water mixtures are arranged in series. Tank A has a capacity of 100 gallons, while Tank B has a capacity of 200 gallons. Each tank is initially filled with water, and 50 lbs of salt is mixed into each tank. Then the following procedure is started:

- Pure water flows into Tank A at a rate of 5 gallons per minute.
- The perfectly mixed solution in Tank A flows out of Tank A and into Tank B at a rate of 5 gallons per minute.
- The perfectly mixed solution in Tank B flows out of Tank B at a rate of 5 gallons per minute.

(a) Write an initial value problem for  $x(t)$ , the amount of salt in Tank A at time  $t$ .

*Answer:*  $x' = -\frac{5}{100}x$ ,  $x(0) = 50$ .

(b) Solve the differential equation which you wrote in part (a).

*Answer:*  $x(t) = 50e^{-t/20}$ .

(c) Explain why the initial value problem describing  $y(t)$ , the amount of salt in Tank B at time  $t$  is

$$\frac{dy}{dt} = \frac{5}{100}x - \frac{5}{200}y, \quad y(0) = 50.$$

*Answer:*  $\frac{dy}{dt}$  is the rate of change of the amount of salt in Tank 2.  $\frac{5}{100}x$  is the contribution to this rate of change coming from the inflow (from Tank 1).  $-\frac{5}{200}y$  is the contribution to this rate of change coming from the outflow.  $y(0) = 50$  is the initial amount of salt in Tank 2.

(d) Using the function  $x(t)$  which you found in part (b), solve the differential equation in part (c).

*Answer:* The initial value problem is  $y' + \frac{1}{40}y = 2.5e^{-t/20}$ ,  $y(0) = 50$ , whose solution is

$$y(t) = 150e^{-t/40} - 100e^{-t/20}.$$

(e) (**Extra Credit**) What is the maximum amount of salt present in Tank B?

*Answer:* We solve  $y'(t) = 0$  and obtain  $T = 40 \log(4/3) \approx 11.5$  minutes. At this time,  $y(T) = \frac{225}{4} = 56.25$  lbs of salt.

2. For which values  $y_0$  is the following statement true? Justify your answer.

The solution  $y(x)$  to the initial value problem

$$y' = y + \sin(y), \quad y(0) = y_0$$

is defined for all real numbers  $x$ , and satisfies  $\lim_{x \rightarrow \infty} y(x) = 0$ .

*Answer:* Only for  $y_0 = 0$ .

The function  $F(y) = y + \sin(y)$  has only one zero, at  $y_0 = 0$ . Computing  $F'(y_0) = 1 + \cos(0) = 2 > 0$  shows that this is an **unstable** equilibrium point. Since there are no other critical points, all solutions starting at any other value of  $y$  diverge from the line  $y = y_0 = 0$ . Thus  $y_0 = 0$  is the only value for which the statement is true.