Math 441 Fall 2005 Quiz #4 Solutions

1. (10 points) (a) Suppose that y_1 and y_2 are linearly independent solutions to a second order ODE

$$y'' + p_1(t)y' + p_0(t)y = 0,$$

where the coefficient functions p_0 and p_1 are continuous on an interval I, and let $W(t) = W(y_1, y_2)(t)$ be the Wronskian of y_1 and y_2. Mark each of the following statements as **always true**, **sometimes true/sometimes false**, or **always false**.

1. $W(t) = 0$ for all $t \in I$. ALWAYS FALSE
2. There exists some $t \in I$ for which $W(t) = 0$. ALWAYS FALSE
3. There exists some $t \in I$ for which $W(t) \neq 0$. ALWAYS TRUE
4. $W(t) \neq 0$ for all $t \in I$. ALWAYS TRUE

(b) Calculate the Wronskian $W(y_1, y_2)$ for $y_1(t) = t^2$ and $y_2(t) = t^3$. At which points is $W(y_1, y_2)$ equal to zero?

Answer: $W(y_1, y_2) = y_1 y_2' - y_1' y_2 = (t^2)(3t^2) - (2t)(t^3) = t^4$. This is equal to zero only when $t = 0$.

(c) Find constants b and c so that the functions y_1 and y_2 from part (b) solve the equation $t^2y'' + bty' + cy = 0$.

Answer: Substitute y_1 and y_2 into the equation to find

$$0 = t^2(2) + bt(2t) + c(t^2) = (2 + 2b + c)t^2$$

and

$$0 = t^2(6t) + bt(3t^2) + c(t^3) = (6 + 3b + c)t^3.$$

In order for this to hold true for all t, we must have $2 + 2b + c = 0$ and $6 + 3b + c = 0$, i.e., $b = -4$ and $c = 6$.

(d) Explain why there is no contradiction between your answers to parts (a), (b) and (c).

Answer: Rewriting the equation in (c) in the form

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = 0$$

reveals the answer; the coefficient functions $p_0(t) = \frac{6}{t^2}$ and $p_1(t) = -\frac{4}{t}$ are not continuous at $t = 0$. Thus there is no guarantee that linear independence of these solutions on an interval containing $t = 0$ must imply that the Wronskian is not equal to zero.

2. (10 points) (a) Solve the initial value problem

$$y'' + 4y' + 3y = 0, \qquad y(0) = 2,\ y'(0) = -1.$$

Answer: The roots of the characteristic equation are $r = -1$ and $r = -3$. The general solution is $y(t) = C_1 e^{-t} + C_2 e^{-3t}$. Using the initial conditions we find

$$C_1 + C_2 = 2 \qquad \text{and} \qquad -C_1 - 3C_2 = -1$$

which gives $C_1 = \frac{5}{2}$ and $C_2 = -\frac{1}{2}$. The solution is

$$y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{-3t}.$$

(b) Use the method of reduction of order to find the general solution to the equation

$$y'' - 6y' + 9y = 0.$$

Answer: $r = 3$ is a double root of the equation. One solution is $y_1(t) = e^{3t}$. To find another solution, we set $y_2(t) = u(t)e^{3t}$ and compute

$$0 = y_2'' + 4y_2' + 3y_2 = (u'' + 6u' + 9u)e^{3t} - 6(u' + 3u)e^{3t} + 9ue^{3t} = u''e^{3t}.$$

Thus $u'' = 0$, so $u(t) = C_1 t + C_2$. Choosing $C_1 = 0$ and $C_2 = 1$ gives $u(t) = 1$ and recovers the original solution $y_2 = y_1$. Choosing $C_1 = 1$ and $C_2 = 0$ gives $u(t) = t$ and the second linearly independent solution $y_2(t) = ty_1(t) = te^{3t}$. The general solution is

$$y(t) = C_1 e^{3t} + C_2 te^{3t}.$$

(c) (**Extra credit**) Let $y(t)$ be the answer you found in part (a). Find constants $r > 0$ and z_0 so that the solution $z(t)$ to the first-order initial value problem

$$z' + rz = 0, \qquad z(0) = z_0$$

satisfies

$$\lim_{t\to\infty} \frac{y(t)}{z(t)} = 1.$$

Answer: $r = 1$ and $z_0 = \frac{5}{2}$.