

## Math 441 Fall 2005 Quiz #6 Solutions

1. (10 points) Consider the equation

$$y'' + xy' + 3y = 0.$$

(a) Find power series representations for two linearly independent solutions to this equation at the point  $x_0 = 0$ . Give the recursion formula for the coefficients, give an explicit form for the general coefficient involving quotients of finite products of integers, and write out the first three nonzero terms in each series explicitly.

*Answer:* We assume a solution of the form  $y(x) = \sum_{n=0}^{\infty} a_n x^n$ . Then  $y'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}$  and  $y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$ . We determine the values of the coefficients from the equation:

$$\begin{aligned} 0 &= y'' + xy' + 3y = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} n a_n x^{n-1} + 3 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 3 a_n x^n \\ &= \sum_{n=0}^{\infty} ((n+2)(n+1) a_{n+2} + (n+3) a_n) x^n. \end{aligned}$$

The coefficients  $a_0$  and  $a_1$  are arbitrary, and further coefficients are determined by the recursion relation

$$a_{n+2} = -\frac{n+3}{(n+2)(n+1)} a_n.$$

Thus

$$a_2 = -\frac{3}{2} a_0, \quad a_3 = -\frac{2}{3} a_1, \quad a_4 = -\frac{5}{12} a_2 = \frac{5}{8} a_0, \quad a_5 = -\frac{3}{10} a_3 = \frac{1}{5} a_1, \quad \dots$$

In general,

$$a_{2m} = (-1)^m \frac{(2m+1)(2m-1)\cdots(5)(3)}{(2m)(2m-2)\cdots(4)(2)(2m-1)(2m-3)\cdots(3)(1)} a_0$$

and

$$a_{2m+1} = (-1)^m \frac{(2m+2)(2m)\cdots(6)(4)}{(2m+1)(2m-1)\cdots(5)(3)(2m)(2m-2)\cdots(4)(2)} a_1.$$

The general solution is

$$\begin{aligned} y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \\ &= a_0 + a_1 x - \frac{3}{2} a_0 x^2 - \frac{2}{3} a_1 x^3 + \frac{5}{8} a_0 x^4 + \frac{1}{5} a_1 x^5 + \cdots \\ &= a_0 \left( 1 - \frac{3}{2} x^2 + \frac{5}{8} x^4 + \cdots \right) + a_1 \left( x - \frac{2}{3} x^3 + \frac{1}{5} x^5 + \cdots \right). \end{aligned}$$

Power series for two linearly independent solutions are

$$y_1(x) = 1 - \frac{3}{2}x^2 + \frac{5}{8}x^4 + \cdots \quad \text{and} \quad y_2(x) = x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + \cdots .$$

(b) On what interval do the series which you found in part (a) converge? Justify your answer without using any of the standard convergence tests (Ratio, Root, etc.).

*Answer:* Since the coefficient functions  $p_1(x) = x$  and  $p_0(x) = 1$  have power series which converge for all values of  $x$ , the solutions  $y_1$  and  $y_2$  have power series which converge for all values of  $x$  also, i.e., the interval of convergence is equal to  $(-\infty, +\infty)$  in each case.

(c) (**Extra credit**) Find explicit expressions for the general coefficients in the two solutions which you found in part (a).

*Answer:* The solution to the recursion relation in part (a) is

$$a_{2m} = (-1)^m \frac{2m+1}{2^m m!} a_0$$

and

$$\begin{aligned} a_{2m+1} &= (-1)^m \frac{2m+2}{(2m+1)(2m-1)\cdots(5)(3)(2)} a_1 \\ &= (-1)^m \frac{(m+1)2^m m!}{(2m+1)!} a_1 \\ &= (-1)^m \frac{2^m (m+1)!}{(2m+1)!} a_1. \end{aligned}$$

Thus the two linearly independent solutions are

$$y_1(x) = \sum_{m=0}^{\infty} \frac{(2m+1)(-\frac{1}{2})^m}{m!} x^{2m}$$

and

$$y_2(x) = \sum_{m=0}^{\infty} \frac{(m+1)!(-2)^m}{(2m+1)!} x^{2m+1}.$$