Math 542 HW #3
due Monday, 9/17

1: #IX.6.31 from Palka. Express the condition $|[z_1, z_2, z_3, z_4]| + |[z_3, z_1, z_2, z_4]| = 1$ directly in terms of distances $|z_j - z_k|$. What theorem from geometry does this condition encode?

2: #IX.6.41 from Palka. The set of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$, $ad-bc \neq 0$,modulo the projective equivalence relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}, \qquad \lambda \in \mathbb{R}, \lambda \neq 0,$$

is denoted by $PSL(2, \mathbb{R})$.

3: #VII.5.16 and VII.5.40 from Palka.

4: #VII.5.24 from Palka.

5: #VII.5.54 from Palka.

6: Let $f : \Omega \to \mathbb{C}$ be an analytic function with $f'(z) \neq 0$ for any $z \in \Omega$.

(a) Let $\gamma : [0, 1] \to \Omega$ be a C^1 curve. Recall that the length of γ is given by $\int_0^1 |\gamma'(t)|\, dt$.

Show that the length of $f \circ \gamma$ is given by $\int_0^1 |f'(\gamma(t))|\, |\gamma'(t)|\, dt$.

(b) The *characteristic function* of a set $E \subset \mathbb{C}$ is $\chi_E(z) = 1$ if $z \in E$ and $\chi_E(z) = 0$ if $z \notin E$. Let E be a subset of Ω so that χ_E is Riemann integrable over Ω. Recall that the area of E is given by $\int\int_\Omega \chi_E(z)\, dx\, dy$, $z = x + iy$.

Show that the area of $f(E)$ is given by $\int\int_\Omega |f'(z)|^2 \chi_E(z)\, dx\, dy$.

(c) Let Γ be a collection of C^1 curves $\gamma : [0, 1] \to \Omega$. Define the *modulus* of Γ to be

$$\operatorname{Mod}\Gamma = \inf_\rho \frac{\int\int_\Omega \rho(z)^2\, dx\, dy}{\inf_{\gamma\in\Gamma} \left(\int_0^1 \rho(\gamma(t))|\gamma'(t)|\, dt \right)^2}$$

where the infimum is taken over all continuous functions $\rho : \Omega \to [0, \infty)$.

Show that $\operatorname{Mod}\Gamma = \operatorname{Mod} f(\Gamma)$ for every such collection Γ, where $f(\Gamma) = \{f \circ \gamma : \gamma \in \Gamma\}$.

Hint: It suffices to prove an inequality. Given $\tilde{\rho} : f(\Omega) \to [0, \infty)$, define $\rho : \Omega \to [0, \infty)$ by $\rho(z) := \tilde{\rho} \circ f(z)\, |f'(z)|$.