Math 542 HW #9
due Friday, 11/16

1: #IX.6.63 from Palka.

2: #IX.6.64 from Palka. The value r is called the *mapping radius* of the domain D with basepoint z_0. Find the mapping radius of $D = \mathbb{C} \setminus (-\infty, 0]$ with basepoint $z_0 = 1$.

3: #IX.6.65 from Palka.

4: #X.4.7 from Palka. You may omit the statement about Legendre's relation. Note that ζ is an *odd* function, since $\mathcal{P}$ is even.

5 (**Extra Credit**): Prove the identity

$$\frac{\mathcal{P}'(z)}{\mathcal{P}(z) - \mathcal{P}(u)} = \zeta(z+u) - 2\zeta(z) + \zeta(z-u), \tag{1}$$

for all distinct $z, u \in \mathbb{C}$, where ζ is the function from the previous problem.[1]

The Laurent series expansions $\mathcal{P}(z) = \frac{1}{z^2} + \sum_{k=2}^{\infty}(2k-1)a_{2k}z^{2k-2}$ and $\zeta(z) = \frac{1}{z} - \sum_{k=2}^{\infty} a_{2k}z^{2k-1}$, where $a_{2k} = \sum_{\omega \in M\setminus\{0\}} \omega^{-2k}$, may be useful. Recall also the equation (proved in class):

$$\mathcal{P}'(z)^2 = 4\mathcal{P}(z)^3 - 60a_4\mathcal{P}(z) - 140a_6. \tag{2}$$

6: (a) Use problem 5 and symmetrization in the variables z, u to show that

$$\zeta(z+u) - \zeta(z) - \zeta(u) = \frac{1}{2}\,\frac{\mathcal{P}'(z) - \mathcal{P}'(u)}{\mathcal{P}(z) - \mathcal{P}(u)}, \qquad \forall\, z, u \in \mathbb{C},\ z \neq u.$$

(b) Deduce from (2) the second-order equation $\mathcal{P}''(z) = 6\mathcal{P}(z)^2 - 30a_4$.

(c) Use (a) and (b) to derive the **addition formula**

$$\mathcal{P}(z+u) = -\mathcal{P}(z) - \mathcal{P}(u) + \frac{1}{4}\left(\frac{\mathcal{P}'(z) - \mathcal{P}'(u)}{\mathcal{P}(z) - \mathcal{P}(u)}\right)^2, \qquad \forall\, z, u \in \mathbb{C},\ z \neq u,$$

and the **duplication formula**

$$\mathcal{P}(2z) = -2\mathcal{P}(z) + \frac{1}{4}\left(\frac{\mathcal{P}''(z)}{\mathcal{P}'(z)}\right)^2, \qquad \forall\, z \in \mathbb{C}.$$

Hint: Differentiate (1), use (2), and symmetrize!

7: #X.4.6 from Palka. More precisely, show that $\mathcal{P}'(\omega_1/2) = \mathcal{P}'(\omega_2/2) = \mathcal{P}'(\omega_3/2) = 0$, where $\omega_3 = \omega_1 + \omega_2$. Show that (2) may be rewritten

$$\mathcal{P}'(z)^2 = 4(\mathcal{P}(z) - e_1)(\mathcal{P}(z) - e_2)(\mathcal{P}(z) - e_3),$$

where $e_j = \mathcal{P}(\omega_j/2)$, and conclude that $e_1 + e_2 + e_3 = 0$.

8: #X.4.16 from Palka.

9: Let $\Omega \subset \mathbb{C}$ be the square with vertices $0, 1, i, 1+i$. Show that the map $f(z) = A\int_i^z \frac{dt}{\sqrt{4(t-e_1)(t-e_2)(t-e_3)}} + B$, $A, B \in \mathbb{C}$, defines a Riemann map from the upper half plane onto Ω which send $e_2 := -e_1, e_3 = 0, e_1, \infty$ to the vertices of Ω (in order).

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[1] Hint for #5: It suffices to prove (1) when u is not a pole of $\mathcal{P}$. Observe that both sides are elliptic functions of z. Expand in Laurent series at $z = 0$ for $|z| < \epsilon$, where ϵ is so small that $|\mathcal{P}(z) - \mathcal{P}(u) - \frac{1}{z^2}| < |\frac{1}{z^2}|$ for $0 < |z| < \epsilon$. Show that $\frac{\mathcal{P}'(z)}{\mathcal{P}(z)-\mathcal{P}(u)} - (\zeta(z+u) - 2\zeta(z) + \zeta(z-u))$ is an entire elliptic function which vanishes at $z = 0$.