

# Mathematics 595 (CAP/TRA) Fall 2005

## 2 Metric and topological spaces

### 2.1 Metric spaces

The notion of a metric abstracts the intuitive concept of “distance”. It allows for the development of many of the standard tools of analysis: continuity, convergence, compactness, etc. Spaces equipped with both a linear (algebraic) structure and a metric (analytic) structure will be considered in the next section. They provide suitable environments for the development of a rich theory of differential calculus akin to the Euclidean theory.

**Definition 2.1.1.** A *metric* on a space  $X$  is a function  $d : X \times X \rightarrow [0, \infty)$  which is symmetric, vanishes at  $(x, y)$  if and only if  $x = y$ , and satisfies the triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y) \quad \text{for all } x, y, z \in X.$$

The pair  $(X, d)$  is called a *metric space*.

Notice that we require metrics to be finite-valued. Some authors consider allow infinite-valued functions, i.e., maps  $d : X \times X \rightarrow [0, \infty]$ . A space equipped with such a function naturally decomposes into a family of metric spaces (in the sense of Definition 2.1.1), which are the equivalence classes for the equivalence relation  $x \sim y$  if and only if  $d(x, y) < \infty$ .

A *pseudo-metric* is a function  $d : X \times X \rightarrow [0, \infty)$  which satisfies all of the conditions in Definition 2.1.1 except that the condition “ $d$  vanishes at  $(x, y)$  if and only if  $x = y$ ” is replaced by “ $d(x, x) = 0$  for all  $x$ ”. In other words, a pseudo-metric is required to vanish along the diagonal  $\{(x, y) \in X \times X : x = y\}$ , but may also vanish for some pairs  $(x, y)$  with  $x \neq y$ .

**Exercise 2.1.2.** Let  $(X, d)$  be a pseudo-metric space. Consider the relation on  $X$  given by  $x \sim y$  if and only if  $d(x, y) = 0$ . Show that this is an equivalence relation, and that the map  $d/\sim$  on the quotient space  $X/\sim$  given by

$$(d/\sim)([x], [y]) := d(x, y)$$

is well-defined. Show that  $d/\sim$  is a metric on  $X/\sim$ . The space  $(X/\sim, d/\sim)$  is called the *quotient metric space* for  $(X, d)$ ; it is a canonically defined metric space associated with the given pseudo-metric space.

Every norm on a vector space defines a translation-invariant metric.

**Definition 2.1.3.** Let  $V$  be a vector space over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{F} = \mathbb{C}$ . A *norm* on  $V$  is a map  $\|\cdot\| : V \rightarrow [0, \infty)$  so that

- (nondegeneracy)  $\|v\| = 0$  if and only if  $v = 0$ ;
- (homogeneity)  $\|av\| = |a| \|v\|$  for all  $a \in \mathbb{F}$  and  $v \in V$ ;

- (sub-additivity)  $\|v + w\| \leq \|v\| + \|w\|$  for all  $v, w \in V$ .

If  $\|\cdot\|$  is a norm on  $V$ , then  $d(v, w) := \|v - w\|$  defines a metric on  $V$ . It is translation invariant in the following sense:  $d(u + v, u + w) = d(v, w)$  for all  $u, v, w \in V$ .

Here are a few examples of metric spaces. The first two are translation invariant metrics arising from norms on (finite-dimensional) vector spaces.

**Examples 2.1.4.** 1. The Euclidean metric on  $\mathbb{R}^n$  is the translation invariant metric associated to the Euclidean norm

$$\|x\|_2 := (x_1^2 + \cdots + x_n^2)^{1/2}, \quad x = (x_1, \dots, x_n).$$

We usually abbreviate  $\|x\|_2 = |x|$  in this case.

2. More generally, let  $1 \leq p \leq \infty$  and define the  $l^p$  norm on  $\mathbb{R}^n$  by

$$\|x\|_p = \begin{cases} (|x_1|^p + \cdots + |x_n|^p)^{1/p}, & 1 \leq p < \infty, \\ \max\{|x_1|, \dots, |x_n|\}, & p = \infty. \end{cases}$$

The translation invariant metric associated with  $\|\cdot\|_p$  is called the  $l^p$ -metric on  $\mathbb{R}^n$ .

3. Let  $X = \{a, b, c\}$  and set  $d(a, b) = 1$ ,  $d(b, c) = 2$ ,  $d(a, c) = 3$ . Can you find a triangle in  $\mathbb{R}^2$  which realizes these distances?
4. (The *French railway*<sup>1</sup> metric) Let  $X = \mathbb{R}^2$  and define

$$d(x, y) = \begin{cases} |x - y|, & \text{if } x = ay \text{ for some } a > 0, \\ |x| + |y|, & \text{otherwise.} \end{cases}$$

**Definition 2.1.5.** Let  $(X, d)$  be a metric space. The (*metric*) *ball* with center  $x \in X$  and radius  $r > 0$  is the set

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Note that this is the **open** ball with this center and radius. Notice also that neither the center nor the radius of a metric ball is necessarily well-defined. For example, in the metric space of Example 2.1.4(3) we have  $B(a, 1.5) = B(b, 1.5) = B(b, 2)$ . For this reason, we have chosen a notation for metric balls which identifies the center and radius explicitly.

**Definition 2.1.6.** Let  $(X, d)$  and  $(Y, d')$  be metric spaces. We say that  $(X, d)$  *embeds isometrically* in  $(Y, d')$  if there exists a function  $i : X \rightarrow Y$  so that  $d'(i(x_1), i(x_2)) = d(x_1, x_2)$  for all  $x_1, x_2 \in X$ .

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<sup>1</sup>or Champaign–Urbana Mass Transit District!

If  $X$  admits an isometric embedding in  $Y$ , then—from the point of view of metric geometry—we may as well consider  $X$  as a subset of  $Y$ : all metric properties of  $X$  are inherited from the ambient space  $Y$ .

The triangle inequality in Examples 2.1.4(1) and (2) is usually called the *Minkowski inequality*. We prove it, as well as the fundamental *Hölder inequality* in the following theorem.

**Theorem 2.1.7 (Hölder and Minkowski inequalities for  $\mathbb{R}^n$ ).** *Let  $v, w \in \mathbb{R}^n$  and let  $1 \leq p, q \leq \infty$  with*

$$\frac{1}{p} + \frac{1}{q} = 1 \quad (2.1.8)$$

*(we interpret  $\frac{1}{\infty} = 0$ ). Then*

$$|v \cdot w| \leq \|v\|_p \|w\|_q \quad (2.1.9)$$

*(where  $v \cdot w$  denotes the usual dot product) and*

$$\|v + w\|_p \leq \|v\|_p + \|w\|_p. \quad (2.1.10)$$

A pair of real numbers  $p$  and  $q$  satisfying (2.1.8) is called a *Hölder conjugate pair*.

**Lemma 2.1.11 (Young's inequality).** *For all Hölder pairs  $p, q$  and all  $s, t > 0$ ,*

$$st \leq \frac{s^p}{p} + \frac{t^q}{q}.$$

There is a simple geometric argument which shows this lemma. Consider the function  $t = s^{p-1}$  in the first quadrant  $\{(s, t) : s, t > 0\}$ . The fact that  $p$  and  $q$  form a Hölder conjugate pair means that the inverse of this function is  $s = t^{q-1}$ . The terms  $\frac{s^p}{p}$  and  $\frac{t^q}{q}$  represent areas of certain regions bounded by the graph of this function and the  $s$ - and  $t$ -axes, respectively. The union of these two regions contains the rectangle  $[0, s] \times [0, t]$ , which proves the lemma.

**Exercise 2.1.12.** Fill in the details in the above argument.

*Proof.* First, we prove Hölder's inequality (2.1.9). By the homogeneity of the norms  $\|\cdot\|_p$  and  $\|\cdot\|_q$ , it suffices to assume that  $\|v\|_p = \|w\|_q = 1$ . If  $v = (v_1, \dots, v_n)$  and  $w = (w_1, \dots, w_n)$ , then

$$|v \cdot w| \leq \sum_{i=1}^n |v_i| |w_i| \leq \sum_i \frac{|v_i|^p}{p} + \frac{|w_i|^q}{q}$$

by Young's inequality

$$= \frac{\|v\|_p^p}{p} + \frac{\|w\|_q^q}{q} = \frac{1}{p} + \frac{1}{q} = 1.$$

To prove Minkowski's inequality (2.1.10), we write

$$\|v + w\|_p^p \leq \sum_{i=1}^n |v_i + w_i|^{p-1} (|v_i| + |w_i|)$$

(why?)

$$\leq \left( \sum_{i=1}^n |v_i + w_i|^p \right)^{(p-1)/p} \left[ \left( \sum_{i=1}^n |v_i|^p \right)^{1/p} + \left( \sum_{i=1}^n |w_i|^q \right)^{1/q} \right]$$

by Hölder's inequality

$$= \|v + w\|_p^{p-1} (\|v\|_p + \|w\|_q).$$

Dividing by  $\|v + w\|_p^{p-1}$  finishes the proof.

□

## 2.2 Maps between metric spaces

In this section we discuss classes of mappings between metric spaces. We begin with the most well-known class, the continuous maps.

**Definition 2.2.13.** A map  $f : (X, d) \rightarrow (Y, d')$  between metric spaces is *continuous* if for every  $x_0 \in X$  and  $\epsilon > 0$  there exists  $\delta > 0$  so that  $d(x, x_0) < \delta$  implies  $d'(f(x), f(x_0)) < \epsilon$ .

Equivalently,  $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon)$ .

The reformulation of continuity in terms of the action of the map on open sets will allow us to extend the definition of continuity to a more general class of spaces (topological spaces) later in this chapter.

**Definition 2.2.14.** An *open set* in a metric space is a (possibly empty) union of (open) balls.

Notice that we make no restriction on the type (disjoint or not) or cardinality of the union; **any** union of open balls defines an open set. The empty union is allowed, i.e., the empty set is understood to be an open set.

**Lemma 2.2.15.** *Let  $U$  be a set in  $X$ . The following are equivalent.*

- (1)  $U$  is an open set,
- (2) For every  $x_0 \in U$  there exists  $r = r(x_0) > 0$  so that  $B(x_0, r) \subset U$ .

*Proof.* Suppose (2) is valid. Then

$$U \subset \bigcup_{x_0 \in U} B(x_0, r(x_0)) \subset U$$

so equality holds and  $U$  is a union of open balls. Thus (1) is true.

For the converse, suppose that  $U$  is an open set and let  $x_0 \in U$ . Then  $x_0 \in B(x_1, \delta) \subset U$  for some  $x_1$  and  $\delta > 0$  and so

$$x_0 \in B(x_0, \delta - d(x_0, x_1)) \subset B(x_1, \delta) \subset U.$$

Thus the condition in (2) holds for this  $x_0$  with  $r(x_0) = \delta - d(x_0, x_1) > 0$ . □

**Proposition 2.2.16.** *Let  $f : (X, d) \rightarrow (Y, d')$  be a function. The following are equivalent.*

- (1)  $f$  is continuous,
- (2) For every open set  $V$  in  $Y$ , the preimage  $U = f^{-1}(V)$  is open in  $X$ .

*Proof.* Suppose (2) is valid and let  $x_0 \in X$ ,  $\epsilon > 0$  be given. Set  $V = B(f(x_0), \epsilon)$ . Since  $V$  is open in  $Y$ ,  $U = f^{-1}(V)$  is open in  $X$ , so (by Lemma 2.2.15)  $U$  is the union of open balls centered at every point of  $U$ . In particular, there is a ball  $B(x_0, \delta) \subset U$ , so  $f(B(x_0, \delta)) \subset f(U) = V = B(f(x_0), \epsilon)$ . Hence  $f$  is continuous.

Now suppose that  $f$  is continuous, and let  $V$  be open in  $Y$ . Let  $x_0 \in U := f^{-1}(V)$  and choose  $\epsilon$  so that  $B(f(x_0), \epsilon) \subset V$ . By hypothesis, there is  $\delta$  so that  $f(B(x_0, \delta)) \subset B(f(x_0), \epsilon) \subset V$ , i.e.,  $B(x_0, \delta) \subset U$ . Since  $x_0$  was an arbitrary point of  $U$ ,  $U$  is open. □

Here are some important classes of examples of continuous functions.

**Definition 2.2.17.** A map  $f : (X, d) \rightarrow (Y, d')$  is *Lipschitz* if there exists  $L < \infty$  so that  $d'(f(x_1), f(x_2)) \leq Ld(x_1, x_2)$  for all  $x_1, x_2 \in X$ .

Every Lipschitz function is continuous; check that the choice  $\delta = \epsilon/L$  works for any point  $x_0 \in X$  and  $\epsilon > 0$ . We often emphasize this by using the term *Lipschitz continuous*.

Lipschitz functions are one of the basic workhorses of modern analysis; their role in the modern theory of analysis on metric spaces is comparable to the role of smooth functions in standard real analysis.

In many situations, it's enough to know that a Lipschitz-type condition holds at each point of  $x_0$  (with Lipschitz constant  $L$  depending on  $x_0$ ).

**Definition 2.2.18.** A map  $f : (X, d) \rightarrow (Y, d')$  is *locally Lipschitz* if for each  $x_0 \in X$  there exists  $r > 0$  so that  $f|_{B(x_0, r)}$  is Lipschitz.

**Examples 2.2.19.** 1. Distance functions are Lipschitz. Fix  $x_0 \in X$  and define  $f : X \rightarrow [0, \infty)$  by  $f(x) = d(x, x_0)$ . Then  $f$  is 1-Lipschitz as a map from  $(X, d)$  to  $\mathbb{R}$  with Euclidean metric. Indeed, for all  $x_1, x_2 \in X$ ,

$$|f(x_1) - f(x_2)| = |d(x_1, x_0) - d(x_2, x_0)| \leq d(x_1, x_2)$$

by the triangle inequality.

2. Every  $C^1$  function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is locally Lipschitz. ( $C^1$  means continuously differentiable:  $f'$  exists and is continuous.) To prove this, we need to use the fact that a continuous (real-valued) function defined on a closed and bounded set in  $\mathbb{R}$  is necessarily bounded. (We will prove a more general version of this fact later on in these notes.) We will show that  $f$  is Lipschitz on any closed and bounded subset of  $\mathbb{R}$ . It's enough to do this for any closed interval  $I$ . By the preceding remark, there exists  $L < \infty$  so that  $|f'(x)| \leq L$  for all  $x \in I$ . For any two points  $a, b \in I$ , there exists  $c \in [a, b]$  so that  $f(b) - f(a) = f'(c)(b - a)$  (Mean Value Theorem). Then

$$|f(b) - f(a)| \leq L|b - a|$$

so  $f|_I$  is Lipschitz.

**Remark 2.2.20.** Hölder functions are generalization of Lipschitz functions. A function  $f : (X, d) \rightarrow (Y, d')$  is *Hölder* (or *Hölder continuous*) with exponent  $\alpha$ ,  $0 < \alpha \leq 1$ , if there exists  $L < \infty$  so that  $d'(f(x_1), f(x_2)) \leq Ld(x_1, x_2)^\alpha$  for all  $x_1, x_2 \in X$ .

**Exercise 2.2.21.** In Problem 4(a) of Homework #3, we observed that the function  $d^\epsilon$  defines a metric on  $X$  whenever  $d$  is a metric on  $X$ . Let  $f : X \rightarrow \mathbb{R}$  be a function. Show that  $f : (X, d) \rightarrow \mathbb{R}$  is  $\epsilon$ -Hölder continuous if and only if  $f : (X, d^\epsilon) \rightarrow \mathbb{R}$  is Lipschitz continuous.

**Definition 2.2.22.** The *modulus of continuity* of a function  $f : (X, d) \rightarrow (Y, d')$  is the function

$$\omega_f(r) = \sup\{d'(f(x), f(y)) : d(x, y) < r\}.$$

Notice that  $\omega_f$  may take on the value  $+\infty$ .

For example, if  $f$  is Lipschitz with constant  $L$ , then  $\omega_f(r) \leq Lr$ . If  $f$  is Hölder with exponent  $\alpha$  and constant  $L$ , then  $\omega_f(r) \leq Lr^\alpha$ .

**Definition 2.2.23.** A function  $f : (X, d) \rightarrow (Y, d')$  is *uniformly continuous* if  $\lim_{r \rightarrow 0} \omega_f(r) = 0$ . Equivalently,  $f$  is uniformly continuous if for all  $\epsilon > 0$  there exists  $\delta > 0$  so that for all  $x, x_0 \in X$ ,  $d(x, x_0) < \delta$  implies  $d'(f(x), f(x_0)) < \epsilon$ .

It's worth reiterating how this differs from the usual notion of continuity; in the definition of uniform continuity the choice of  $\delta$  must be made independently of the point  $x_0$ .

To conclude this section, we discuss the Lipschitz equivalence of the  $l^p$  metrics on  $\mathbb{R}^n$ .

Two metrics  $d$  and  $d'$  on a space  $X$  are called *Lipschitz equivalent* if the identity map from  $(X, d)$  to  $(X, d')$  and its inverse are Lipschitz continuous. In a similar manner, we define the notions of *Hölder equivalence*, *uniform equivalence*, etc. All of these notions are equivalence relations on the collection of all metrics on  $X$ .

**Proposition 2.2.24.** Fix  $n \in \mathbb{N}$ . The  $l^p$  metrics  $\|\cdot\|_p$ ,  $1 \leq p \leq \infty$  on  $\mathbb{R}^n$  are all mutually Lipschitz equivalent.

*Proof.* It's enough to show that  $\|\cdot\|_p$  and  $\|\cdot\|_\infty$  are Lipschitz equivalent. In fact, we will show that

$$n^{-1/p} \|x\|_p \leq \|x\|_\infty \leq \|x\|_p \tag{2.2.25}$$

for all  $x \in \mathbb{R}^n$ . Recall that  $\|x\|_\infty = \max_i |x_i|$  and  $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$ , where  $x = (x_1, \dots, x_n)$ . The right hand inequality in (2.2.25) is obvious. For the left hand inequality, compute

$$\|x\|_p^p = \sum_i |x_i|^p \leq n \|x\|_\infty^p$$

and take  $p$ th roots. □

**Remark 2.2.26.** In fact, any two norms on a finite-dimensional vector space define Lipschitz equivalent translation invariant metrics.

**Exercise 2.2.27.** From (2.2.25) it easily follows that

$$n^{-1/q} \|x\|_q \leq \|x\|_p \leq n^{1/p} \|x\|_q$$

for all  $1 \leq p, q \leq \infty$  and all  $x \in \mathbb{R}^n$ . However, these are not the best constants for the Lipschitz equivalence of  $(\mathbb{R}^n, \|\cdot\|_p)$  and  $(\mathbb{R}^n, \|\cdot\|_q)$ . Show that

$$\|x\|_q \leq \|x\|_p \leq n^{1/p-1/q} \|x\|_q$$

for all  $1 \leq p \leq q \leq \infty$  and all  $x \in \mathbb{R}^n$ . (Hint: one direction follows from Hölder's inequality, using the Hölder conjugate pairs  $\frac{q}{p}$  and  $\frac{q}{q-p}$ . For the other direction, compare Exercise 4(a) on Homework #3.)

## 2.3 Sequences, complete metric spaces, and completions

**Definition 2.3.28.** A *sequence* in a set  $X$  is a function  $\sigma : \mathbb{N} \rightarrow X$ . We usually write  $\sigma(n) = x_n$ , and denote the full sequence by  $(x_n)$ .

**Definition 2.3.29.** A sequence  $(x_n)$  in a metric space  $(X, d)$  *converges* to  $x_\infty$  if for all  $\epsilon > 0$  there exists  $N < \infty$  so that

$$d(x_n, x_\infty) < \epsilon$$

for all  $n \geq N$ . A sequence  $(x_n)$  is *convergent* if it converges to some point. We will use the notations

$$\lim_{n \rightarrow \infty} x_n = x_\infty \quad (\text{or } \lim_n x_n = x_\infty)$$

or

$$x_n \rightarrow x_\infty \quad (n \rightarrow \infty)$$

interchangeably.

**Examples 2.3.30.** (1) Let  $(X, d)$  be  $\mathbb{R}$  with the Euclidean metric. Then  $(\frac{1}{n}) \rightarrow 0$ .

(2) Let  $(X, d)$  be  $\mathbb{R}^m$  with the Euclidean metric. Then  $(\frac{1}{n}e_{n \bmod m}) \rightarrow (0, \dots, 0)$ . Here  $n \bmod m$  denotes the remainder upon dividing  $n$  by  $m$ .

**Exercise 2.3.31.** Let  $d$  be the discrete metric on a set  $X$ :  $d(x, y) = 1$  if  $x \neq y$  and  $d(x, x) = 0$  for all  $x$ . Which sequences in  $X$  converge? What do they converge to?

**Proposition 2.3.32.** Let  $f : (X, d) \rightarrow (Y, d')$  be a function. The following are equivalent.

(1)  $f$  is continuous,

(2) For every convergent sequence  $(x_n)$  in  $X$ ,  $(f(x_n))$  is convergent in  $Y$  and  $f(\lim_n x_n) = \lim_n f(x_n)$ .

*Proof.* First, assume that  $f$  is continuous and let  $(x_n)$  be a convergent sequence in  $X$  converging to  $x_\infty$ . Fix  $\epsilon > 0$ . Then there exists  $\delta > 0$  so that  $d(x, x_\infty) < \delta$  implies  $d'(f(x), f(x_\infty)) < \epsilon$ . On the other hand, there exists  $N < \infty$  so that  $d(x_n, x_\infty) < \delta$  for all  $n \geq N$ . Hence  $d'(f(x_n), f(x_\infty)) < \epsilon$  for all  $n \geq N$ , which shows that  $\lim_n f(x_n) = f(x_\infty)$ .

We prove the other direction by contradiction. Suppose  $f$  is not continuous. Then there exists  $x_0 \in X$  and  $\epsilon > 0$  so that for all  $\delta > 0$  there is  $x = x(\epsilon, \delta)$  so that  $d(x_0, x) < \delta$  and  $d'(f(x_0), f(x)) \geq \epsilon$ . We apply this for  $\delta = \frac{1}{k}$ ,  $k \in \mathbb{N}$  to produce a sequence  $(x_k)$ ,  $x_k = x(\epsilon, \frac{1}{k})$  so that  $d(x_0, x_k) < \frac{1}{k}$  and  $d'(f(x_0), f(x_k)) \geq \epsilon$ . Then  $x_k \rightarrow x_0$  but  $f(x_k) \not\rightarrow f(x_0)$ , which contradicts the assumption.  $\square$

Cauchy sequences are sequences in a metric space which “should” converge. A complete metric space is a space in which every sequence which “should” converge actually does converge.

**Lemma 2.3.33.** If  $(x_n)$  is a convergent sequence in a metric space  $(X, d)$ , then for all  $\epsilon > 0$  there exists  $N < \infty$  so that  $d(x_n, x_m) < \epsilon$  whenever  $n, m \geq N$ .

*Proof.* Suppose that  $x_n \rightarrow x_\infty$ . Given  $\epsilon > 0$  choose  $N$  so that  $d(x_n, x_\infty) < \frac{1}{2}\epsilon$  whenever  $n \geq N$ . If  $n, m \geq N$ , then  $d(x_n, x_m) \leq d(x_n, x_\infty) + d(x_\infty, x_m) < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$ .  $\square$

**Definition 2.3.34.** A sequence  $(x_n)$  is *Cauchy* if the conclusion of Lemma 2.3.33 holds. A metric space  $(X, d)$  is *complete* if every Cauchy sequence is convergent.

**Examples 2.3.35.** (1)  $\mathbb{R}$  is complete;  $\mathbb{Q}$  is not complete. (Why?)

(2)  $(\mathbb{R}^n, \|\cdot\|_p)$  is complete for any  $n$  and any  $1 \leq p \leq \infty$ . Let's check this for  $n = p = 2$ . Suppose that  $((x_n, y_n))$  is Cauchy in  $\mathbb{R}^2$  (Euclidean metric). Since  $|x_n - x_m| \leq \|(x_n, y_n) - (x_m, y_m)\|_2$  and  $|y_n - y_m| \leq \|(x_n, y_n) - (x_m, y_m)\|_2$ ,  $(x_n)$  and  $(y_n)$  are Cauchy in  $\mathbb{R}$ . Since  $\mathbb{R}$  is complete, there exist points  $x_\infty, y_\infty \in \mathbb{R}$  so that  $x_n \rightarrow x_\infty$  and  $y_n \rightarrow y_\infty$ . We claim that  $(x_n, y_n) \rightarrow (x_\infty, y_\infty)$ . Given  $\epsilon > 0$  choose  $N_1$  and  $N_2$  so that  $|x_n - x_\infty| < \epsilon/\sqrt{2}$  for all  $n \geq N_1$  and  $|y_n - y_\infty| < \epsilon/\sqrt{2}$  for all  $n \geq N_2$ . Let  $N$  be the maximum of  $N_1$  and  $N_2$ . If  $n \geq N$  then

$$\|(x_n, y_n) - (x_m, y_m)\|_2 = (|x_n - x_m|^2 + |y_n - y_m|^2)^{1/2} < (2(\epsilon/\sqrt{2})^2)^{1/2} = \epsilon.$$

This shows that  $(x_n, y_n) \rightarrow (x_\infty, y_\infty)$ , as desired.

**Lemma 2.3.36.** *Uniformly continuous functions maps Cauchy sequences to Cauchy sequences.*

*Proof.* Let  $f : (X, d) \rightarrow (Y, d')$  be uniformly continuous and let  $(x_n)$  in  $X$  be a Cauchy sequence. Let  $\epsilon > 0$  and choose  $\delta > 0$  as in the definition of uniform continuity. Since  $(x_n)$  is Cauchy there exists  $N$  so that  $d(x_n, x_m) < \delta$  whenever  $n, m \geq N$ . Then  $d'(f(x_n), f(x_m)) < \epsilon$  for all such  $n, m$ , so  $(f(x_n))$  is Cauchy.  $\square$

**Definition 2.3.37.** A set  $C \subset X$  is *closed* if  $X \setminus C$  is open.

**Lemma 2.3.38.**  *$C$  is closed if and only if the following holds:*

$$\text{Whenever } (x_n) \subset C \text{ is a convergent sequence, then } \lim_n x_n \in C. \quad (2.3.39)$$

*Proof.* Suppose that  $C$  is closed and that  $x_\infty = \lim_n x_n$  with  $(x_n) \subset C$  and  $x_\infty \notin C$ . Since  $X \setminus C$  is open, there exists  $\delta > 0$  so that  $B(x_\infty, \delta) \subset X \setminus C$ . This implies that  $d(x_\infty, x_n) \geq \delta$  for all  $n$ , which contradicts the assumption that  $x_n \rightarrow x_\infty$ .

Conversely, suppose that  $C$  is not closed, but (2.3.39) holds. Then  $X \setminus C$  is not open, so there exists  $x_\infty \in X \setminus C$  so that for all  $\delta > 0$ ,  $B(x_\infty, \delta) \cap C \neq \emptyset$ . Apply this for  $\delta = \frac{1}{k}$ ,  $k \in \mathbb{N}$  to produce a sequence  $(x_k) \subset C$  with  $d(x_\infty, x_k) < \frac{1}{k}$ . Thus  $x_k \rightarrow x_\infty$ . Hence  $(x_k)$  is a convergent sequence in  $C$ , but  $x_\infty = \lim_k x_k \in X \setminus C$ , which contradicts (2.3.39).  $\square$

Arbitrary intersections of closed sets are still closed, since arbitrary unions of open sets are still open.

**Definition 2.3.40.** A set  $A \subset X$  is *dense* if every nonempty open set  $O$  in  $X$  has nonempty intersection with  $A$ .

For example, the rationals  $\mathbb{Q}$  are dense in the reals  $\mathbb{R}$ .

**Remark 2.3.41.** By Lemma 2.2.15,  $A$  is dense in  $X$  if and only if every open ball  $B(x_0, r)$  has nonempty intersection with  $A$ . Choosing  $r = \frac{1}{k}$  for each  $k \in \mathbb{N}$ , we obtain that  $A$  is dense in  $X$  if and only if every point in  $X$  is the limit of a sequence from  $A$ .

**Definition 2.3.42.** Let  $A \subset X$ . The *closure* of  $A$  is  $\bar{A} = \bigcap_{C \text{ closed: } A \subset C} C$ .

Being an intersection of closed sets, the closure of a set is a closed set. Clearly  $A \subset \bar{A}$  for any  $A$ .

**Lemma 2.3.43.** For any set  $A \subset X$ ,  $A$  is dense in  $\bar{A}$ .

*Proof.* Suppose not. Then there exists  $O$  open in  $X$  so that  $O \cap \bar{A} \neq \emptyset$  but  $O \cap A = \emptyset$ . Let  $C = X \setminus O$ ;  $C$  is a closed set. Since  $O \cap A = \emptyset$ ,  $A \subset C \cap \bar{A}$ . Since  $O \cap \bar{A} \neq \emptyset$ ,  $C \cap \bar{A} \subsetneq \bar{A}$ . Since  $C \cap \bar{A}$  is closed, we have contradicted the definition of  $\bar{A}$ .  $\square$

We now turn to the notion of the completion of a (non-complete) metric space. We give the traditional definition via equivalence classes of Cauchy sequences. In the next section, we give an alternate description of the completion which is more algebraic.

**Definition 2.3.44.** Given two sequences  $\bar{x} = (x_n)$  and  $\bar{y} = (y_n)$  in a set  $X$ , we can define a new sequence  $\bar{x} \vee \bar{y} = (z_n)$  by setting  $z_{2n-1} = x_n$  and  $z_{2n} = y_n$  for all  $n \in \mathbb{N}$ . This interleaves the elements of the sequences  $\bar{x}$  and  $\bar{y}$ . We call  $\bar{x} \vee \bar{y}$  the *join* of  $\bar{x}$  and  $\bar{y}$ .

We now define an equivalence relation on the set of all Cauchy sequences in a metric space  $(X, d)$ : two such sequences  $\bar{x}$  and  $\bar{y}$  are equivalent if and only if  $\bar{x} \vee \bar{y}$  is Cauchy. We denote the set of equivalence classes for this equivalence relation by  $\bar{X}$ , and call it the *completion* of  $X$ .

Notice that the notation which we use for the completion is the same as that which we used for the closure of a set. After Theorem 2.3.47 this will cause no confusion.

**Exercise 2.3.45.** Show that the relation in this definition is in fact an equivalence relation.

Note that there is a natural embedding  $\iota$  of  $X$  into  $\bar{X}$  given by  $x \mapsto (x, x, x, \dots)$ . We now want to equip the completion with a metric so that this becomes an isometric embedding.

**Lemma 2.3.46.** The function  $\bar{d} : \bar{X} \times \bar{X} \rightarrow [0, \infty)$  given by  $\bar{d}([\bar{x}], [\bar{y}]) = \lim_n d(x_n, y_n)$ ,  $\bar{x} = (x_n)$ ,  $\bar{y} = (y_n)$ , is well-defined.

*Proof.* It's enough to show that if  $\bar{x} = (x_n) \sim \bar{z} = (z_n)$  and  $\bar{y} = (y_n)$  are Cauchy sequences, then  $\lim_n d(x_n, y_n) = \lim_n d(z_n, y_n)$ . By the triangle inequality, this will follow if we can show  $\lim_n d(x_n, z_n) = 0$ . Since  $\bar{x} \sim \bar{z}$ ,  $\bar{x} \vee \bar{z} =: \bar{w} = (w_n)$  is a Cauchy sequence. In other words, for all  $\epsilon > 0$  there exists  $N$  so that  $m, n \geq N$  implies  $d(w_n, w_m) < \epsilon$ . We may assume that  $N = 2M + 1$  is odd. From the definition of the join, we have

$$d(x_n, x_m) < \epsilon, \quad d(x_n, z_m) < \epsilon, \quad \text{and} \quad d(z_n, z_m) < \epsilon$$

for all  $n, m \geq M$ . In particular,  $d(x_n, z_n) < \epsilon$  for all  $n \geq M$ . Thus  $\lim_n d(x_n, z_n) = 0$ .  $\square$

Another way to think about this is as a quotient metric space arising from a pseudometric space (see Exercise 2.1.2). In fact, the collection of all Cauchy sequences in  $(X, d)$ , with the metric  $\bar{d}$  from the lemma, forms a pseudometric space, and  $(\bar{X}, \bar{d})$  is precisely the quotient metric space.

**Theorem 2.3.47.** 1.  $(\bar{X}, \bar{d})$  is a complete metric space,

2. the embedding  $\iota : X \rightarrow \bar{X}$  given above is an isometric embedding so that  $\iota(X)$  is dense in  $\bar{X}$ .

The proof is an abstract version of the classical argument used to construct the real numbers as a completion of the rational numbers. In the next section, we will give an alternate description of the completion of a metric space  $(X, d)$  as a subset of a certain space of functions defined on  $X$ .

*Proof.* We omit the proof that  $\bar{d}$  is a metric on  $\bar{X}$ . The completeness of  $(\bar{X}, \bar{d})$  will follow once we prove part 2, using Lemma 2.3.48. (Apply the lemma with  $A = \iota(X)$  and  $Z = \bar{X}$ , and observe that if  $(x_n)$  is a Cauchy sequence in  $X$ , then  $([\iota(x_n)])$  converges to  $[(x_1, x_2, x_3, \dots)]$ .)

The fact that  $\iota$  is an isometric embedding is obvious, since  $\iota(x)$  is the constant sequence  $(x, x, x, \dots)$  and  $\bar{d}([\bar{x}], [\bar{y}]) = \lim_n d(x_n, y_n)$ .

Let  $\epsilon > 0$  and let  $[\bar{x}] \in \bar{X}$ . Since  $\bar{x} = (x_1, x_2, \dots)$  is Cauchy, there exists  $N$  so that  $d(x_n, x_m) < \epsilon$  for all  $n, m \geq N$ . Let  $oy = (x_N, x_N, x_N, \dots) = \iota(x_N)$ . Then

$$\bar{d}([\bar{x}], [\bar{y}]) = \lim_n d(x_n, y_n) = \lim_n d(x_n, x_N) \leq \epsilon,$$

so  $B([\bar{x}], \epsilon) \cap \iota(X) \neq \emptyset$ . By Remark 2.3.41,  $\iota(X)$  is dense in  $\bar{X}$ .  $\square$

**Lemma 2.3.48.** Let  $(Z, d)$  be a metric space, and let  $A \subset Z$  be a dense subset. Suppose that every Cauchy sequence in  $A$  converges to some element of  $Z$ . Then  $(Z, d)$  is complete.

*Proof.* The proof is a classical diagonal argument in the spirit of Cantor. Suppose that  $(z_m)$  is a Cauchy sequence in  $Z$ . Since  $A$  is dense in  $Z$ , each  $z_m$  is the limit of a Cauchy sequence  $(a_{mn})$  in  $A$ . We now make a diagonal construction. For each  $m$ , choose  $N_m$  so that  $d(a_{mn}, a_{mn'}) < 2^{-m}$  whenever  $n, n' \geq N_m$ . Consider the sequence  $(a_{mN_m})$ . Given  $\epsilon > 0$ , pick  $M$  so that  $2^{-M} < \epsilon/3$  and  $d(z_m, z_{m'}) < \epsilon/3$  whenever  $m, m' \geq M$ . Since  $d(a_{mN_m}, z_m) = \lim_n d(a_{mN_m}, a_{mn}) \leq 2^{-M} < \epsilon/3$ , we conclude that

$$d(a_{mN_m}, a_{m'N_{m'}}) \leq d(a_{mN_m}, z_m) + d(z_m, z_{m'}) + d(z_{m'}, a_{m'N_{m'}}) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

Thus  $(a_{mN_m})$  is a Cauchy sequence in  $A$ . By assumption, it converges to some element  $z \in Z$ . It's easy to check (exercise) that  $\lim_m z_m = z$ . Hence  $(Z, d)$  is complete.  $\square$

**Proposition 2.3.49.** *Let  $A$  be dense in  $(X, d)$  and let  $f : A \rightarrow (Y, d')$  be a uniformly continuous function valued in a complete metric space. Then there exists a unique uniformly continuous function  $\bar{f} : X \rightarrow Y$  so that  $\bar{f}|_A = f|_A$ . Moreover, the modulus of continuity of the extension agrees with that of the original function:*

$$\omega_{\bar{f}} = \omega_f. \quad (2.3.50)$$

*Proof.* By Remark 2.3.41, each  $x \in X$  is the limit of a sequence  $(a_n)$  from  $A$ . Define  $\bar{f}(x) = \lim_n f(a_n)$ . Notice that the limit exists since  $(a_n)$  is Cauchy, hence (by Lemma 2.3.36)  $(f(a_n))$  is Cauchy, hence (since  $(Y, d')$  is complete)  $(f(a_n))$  is a convergent sequence. We need to show that the limit is independent of the choice of sequence. Let  $(b_n)$  be another sequence in  $A$  with limit  $x$  and let  $\epsilon > 0$ . Choose  $\delta > 0$  so that  $d(x_1, x_2) < \delta$ ,  $x_1, x_2 \in A$ , implies  $d(f(x_1), f(x_2)) < \epsilon$ . Choose  $N$  so that

$$d(a_n, x) < \frac{\delta}{2}$$

and

$$d(b_m, x) < \frac{\delta}{2}$$

for all  $n, m \geq N$ . Thus  $d(a_n, b_m) < \delta$  so  $d(f(a_n), f(b_m)) < \epsilon$  for all  $n, m \geq N$ . Sending  $n \rightarrow \infty$  gives  $d(\bar{f}(x), f(b_m)) < \epsilon$  for all  $m \geq N$ . Thus  $\lim_m f(b_m) = \bar{f}(x)$ . If  $x \in A$  we may choose  $(a_n)$  to be the constant sequence  $a_n = x$ , which shows that  $\bar{f}|_A = f|_A$ . Finally, let us show (2.3.50). Given  $x_1, x_2 \in X$  with  $d(x_1, x_2) < \delta$ , we may choose  $(a_n)$  and  $(b_n)$  in  $A$  converging to  $x_1$  and  $x_2$  respectively. We may as well assume that

$$\max\{d(a_n, x_1), d(b_n, x_2)\} < \frac{\delta - d(x_1, x_2)}{2}$$

for all  $n$ . Then, for all  $n \in \mathbb{N}$ , we have

$$d(a_n, b_n) \leq d(a_n, x_1) + d(x_1, x_2) + d(x_2, b_n) < \delta$$

so

$$d'(\bar{f}(x_1), \bar{f}(x_2)) = \lim_n d'(f(a_n), f(b_n)) \leq \omega_f(\delta).$$

Thus  $\omega_{\bar{f}} \leq \omega_f$ ; the other direction is trivial.  $\square$

**Theorem 2.3.51.** *The completion of a metric space is unique, in the following sense. If  $(Y, d')$  is a complete metric space and  $\iota' : (X, d) \rightarrow (Y, d')$  is an isometric embedding so that  $\iota'(X)$  is dense in  $Y$ , then there exists an isometry  $I$  of  $(\bar{X}, \bar{d})$  onto  $(Y, d')$  so that  $I \circ \iota = \iota'$ .*

*Proof.* The map  $\iota' \circ \iota^{-1} : \iota(X) \rightarrow Y$  is an isometric embedding, hence 1-Lipschitz. By Proposition 2.3.49, there exists a 1-Lipschitz map  $I : (\bar{X}, \bar{d}) \rightarrow (Y, d')$  so that  $I \circ \iota = \iota'$ . Similarly  $\iota \circ (\iota')^{-1} : \iota'(X) \rightarrow \bar{X}$  is an isometric embedding, hence 1-Lipschitz, so there exists a 1-Lipschitz map  $J : (Y, d) \rightarrow (\bar{X}, \bar{d})$  so that  $J \circ \iota' = \iota$ . Composing these formulas gives  $J \circ I = \text{id}$  on  $\iota(X)$  and  $I \circ J = \text{id}$  on  $\iota'(X)$ . This implies that  $I|_{\iota(X)}$  is an isometry onto its image, hence  $I$  is an isometry. Similarly  $J$  is an isometry; since  $J$  is the inverse of  $I$  we conclude that  $I$  is onto.  $\square$

## 2.4 Function spaces

In this section, we introduce the notion of function spaces, i.e., spaces of functions between metric spaces, which we equip with various metrics. The goal of this section is to prove a theorem characterizing the completion of a metric space  $(X, d)$  as a subset of a particular function space over  $X$ .

The basic function space which we consider is the space of all continuous functions between two metric spaces.

**Definition 2.4.52.** For metric spaces  $(X, d)$  and  $(X', d')$ , let

$$C(X, Y) = \{f : X \rightarrow Y \mid f \text{ is continuous}\}$$

be the space of all continuous functions from  $(X, d)$  to  $(Y, d')$ .

A “natural” distance function to consider on this space is the maximum metric

$$d_\infty(f, g) = \sup_{x \in X} d'(f(x), g(x)).$$

This satisfies all of the properties of a metric except that it may take on the value  $+\infty$ . For example, the distance between the functions  $f(x) = 1$  and  $g(x) = x$  ( $X = Y = \mathbb{R}$ ) is  $+\infty$ . For this reason, we will most often work with the following subspace of  $C(X, Y)$ .

**Definition 2.4.53.** Let  $(X, d)$  and  $(Y, d')$  be metric spaces and let  $y_0 \in Y$ . We define

$$C_b(X, Y) = \{f \in C(X, Y) : \sup_{x \in X} d'(f(x), y_0) < \infty\}$$

to be the space of all bounded continuous functions from  $(X, d)$  to  $(Y, d')$ .

Notice that, while the definition of  $C_b(X, Y)$  depends on the choice of  $y_0$ , the set of functions so obtained remains the same if  $y_0$  is changed. Indeed, if  $y_1 \in Y$  is another point, then

$$\left| \sup_{x \in X} d'(f(x), y_0) - \sup_{x \in X} d'(f(x), y_1) \right| \leq d'(y_0, y_1) < \infty.$$

**Exercise 2.4.54.** Show that the maximum metric  $d_\infty$  is a metric on  $C_b(X, Y)$ .

**Example 2.4.55.** Let  $X = \{a, b\}$  be a space of two points, with the discrete metric  $d(a, b) = 1$ , and let  $Y = \mathbb{R}$ . Then every function from  $X$  to  $Y$  is continuous and bounded. In fact, we can identify  $C_b(X, \mathbb{R})$  with  $\mathbb{R}^2$  by identifying  $f$  with its range  $(f(a), f(b))$ . In this case,  $(C_b(X, \mathbb{R}), d_\infty)$  is isometric with  $(\mathbb{R}^2, \|\cdot\|_\infty)$  (the  $l^\infty$  metric).

**Remark 2.4.56.** Suppose that  $(Y, d') = (V, \|\cdot\|)$  is a vector space over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{F} = \mathbb{C}$  equipped with a translation invariant metric arising from a norm. Then  $C_b(X, Y)$  has the structure of a vector space, and  $d_\infty$  is a translation invariant metric arising from a norm on  $C_b(X, Y)$ . The vector space structure on  $C_b(X, Y)$  is given by

$$(\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x),$$

where  $f, g \in C_b(X, Y)$ ,  $x \in X$  and  $\alpha, \beta \in \mathbb{F}$ , while the norm on  $C_b(X, Y)$  is given by

$$\|f\| = \sup_{x \in X} \|f(x)\|.$$

**Exercise 2.4.57.** Verify the claims in the preceding remark.

A special case in the above definitions is  $(Y, d') = \mathbb{R}$ , i.e., the case of real-valued functions on  $(X, d)$ . In this case, we abbreviate

$$C(X, \mathbb{R}) = C(X), \quad \text{and} \quad C_b(X, \mathbb{R}) = C_b(X).$$

We are now ready to give an alternate description of the completion of a metric space as a subset of the function space  $C_b(X)$ . The embedding of  $(X, d)$  in  $C_b(X)$  in (2.4.60) is due to Kuratowski.

We need the following elementary lemma.

**Lemma 2.4.58.** *Every closed subset of a complete metric space is complete (in the induced metric).*

*Proof.* Let  $(X, d)$  be a complete metric space, and let  $C$  be a closed subset of  $X$ . Let  $(x_n)$  be a Cauchy sequence in  $C$ . Since  $(X, d)$  is complete,  $(x_n)$  converges to a limit  $x_\infty$  in  $X$ . Since  $C$  is closed,  $x_\infty \in C$ . Thus  $(C, d)$  is complete.  $\square$

**Theorem 2.4.59.** *Let  $(X, d)$  be a metric space and  $(Y, d')$  be a complete metric space. Then  $(C_b(X, Y), d_\infty)$  is a complete metric space.*

*Proof.* Suppose that  $(f_n)$  is a Cauchy sequence of bounded continuous functions with respect to  $d_\infty$ . Then  $(f_n(x))$  is a Cauchy sequence in  $Y$  for each  $x \in X$ ; by the completeness of  $Y$  there exists  $f(x) = \lim_n f_n(x)$ . We must show that  $f \in C_b(X, Y)$ , and that  $d_\infty(f_n, f) \rightarrow 0$ .

Let  $\epsilon > 0$ , and choose  $N$  so that

$$d_\infty(f_n, f_m) < \frac{\epsilon}{6}$$

for all  $n, m \geq N$ . For each  $x \in X$  we may choose  $m = m(x) \geq N$  so that

$$d'(f_m(x), f(x)) < \frac{\epsilon}{6}.$$

Then

$$d'(f_n(x), f(x)) < \frac{\epsilon}{3}$$

for all  $n \geq N$  and  $x \in X$ , so

$$\sup_{n \geq N} d_\infty(f_n, f) < \frac{\epsilon}{3}.$$

Thus  $d_\infty(f_n, f) \rightarrow 0$ . Moreover, if  $\delta > 0$  is chosen so that  $d(x, x_1) < \delta$  implies  $d'(f_N(x), f_N(x_1)) < \epsilon/3$ , then

$$\begin{aligned} d'(f(x), f(x_1)) &\leq d'(f(x), f_N(x)) + d'(f_N(x), f_N(x_1)) + d'(f_N(x_1), f(x_1)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Thus  $f$  is continuous. Finally,

$$\sup_x d'(f(x), y_0) \leq \sup_x d'(f_N(x), y_0) + d_\infty(f, f_N) < \infty$$

so  $f \in C_b(X, Y)$ .  $\square$

We now fix a point  $x_0 \in X$  and consider the map  $e$  from  $(X, d)$  to  $(C(X), d_\infty)$  given by

$$e(x)(y) = d(x, y) - d(x_0, y). \quad (2.4.60)$$

**Proposition 2.4.61.** *The map  $e$  is an isometric embedding of  $(X, d)$  into  $(C_b(X), d_\infty)$ .*

*Proof.* First, we check that  $e(x)$  is a bounded function for each  $x \in X$ . This is immediate from the triangle inequality:

$$\sup_{y \in X} |e(x)(y)| = \sup_{y \in X} |d(x, y) - d(x_0, y)| \leq d(x, x_0) < \infty.$$

Next, we check that  $e$  is an isometry. On the one hand, we have

$$d_\infty(e(x_1), e(x_2)) = \sup_y |e(x_1)(y) - e(x_2)(y)| = \sup_y |d(x_1, y) - d(x_2, y)| \leq d(x_1, x_2).$$

On the other hand,

$$d_\infty(e(x_1), e(x_2)) \geq |e(x_1)(x_2) - e(x_2)(x_2)| = d(x_1, x_2).$$

□

**Theorem 2.4.62.**  *$(\overline{e(X)}, d_\infty)$  is isometric with the completion  $(\overline{X}, \overline{d})$  (defined in the previous section).*

*Proof.* By Lemma 2.4.58, the closure of  $e(X)$  in  $C_b(X)$  is complete with respect to the metric  $d_\infty$ . Now the conclusion follows from the uniqueness up to isometry of the completion  $(\overline{X}, \overline{d})$ . □

One disadvantage of the Kuratowski embedding is that the receiving space  $C_b(X)$  depends on the source space  $X$ .

**Definition 2.4.63.** A metric space is *separable* if it admits a countable dense subset.

For example,  $\mathbb{R}^n$  is separable for each  $n$ .

**Definition 2.4.64.** The space  $\ell^\infty$  consists of all bounded sequences of real numbers, e.g.,  $x = (x_n) \in \ell^\infty$  if and only if  $x_n \in \mathbb{R}$  for all  $n$  and  $\sup_n |x_n| < \infty$ . We equip  $\ell^\infty$  with the *maximum metric*

$$d_\infty(x, y) = \sup_n |x_n - y_n|, \quad x = (x_n), y = (y_n).$$

**Exercise 2.4.65.** Prove that  $(\ell^\infty, d_\infty)$  is complete but not separable.

The embedding in the following theorem is due to Fréchet. It exhibits a universal embedding space for separable metric spaces.

**Theorem 2.4.66.** *Every separable metric space admits an isometric embedding in  $(\ell^\infty, d_\infty)$ .*

Observe that the target space in this theorem is universal, i.e., does not depend on the source. One potential criticism, however, is that this target is itself not separable. There do exist separable metric spaces which contain isometric copies of every separable metric space. We do not discuss this in these notes.

*Proof.* Let  $\{x_1, x_2, \dots\}$  be a countable dense subset of  $X$ . Define  $i : X \rightarrow \ell^\infty$  by

$$i(x)_n = d(x, x_n) - d(x_0, x_n).$$

Observe that

$$\|i(x) - i(y)\|_\infty = \sup_n |d(x, x_n) - d(y, x_n)| \leq d(x, y)$$

for all  $x, y \in X$ . Conversely, let  $x, y \in X$ . There exists a subsequence  $x_{n_1}, x_{n_2}, \dots$  which converges to  $y$ . Then

$$d(x, y) = \lim_{k \rightarrow \infty} |d(x, x_{n_k}) - d(y, x_{n_k})| \leq \sup_n |d(x, x_n) - d(y, x_n)| = \|i(x) - i(y)\|_\infty.$$

Thus  $i$  is an isometric embedding. □

The embedding in Theorem 2.4.66 is due to Frechet.

We will discuss other embedding theorems for metric spaces in Part III of these notes.

## 2.5 A famous example

In this section we want to identify the completion of  $C([0, 1])$  with respect to the metric

$$d_1(f, g) = \|f - g\|_1 = \int_0^1 |f(t) - g(t)| dt.$$

The integration here is in the sense of Riemann.

Let us denote the completion  $\overline{C([0, 1])}$  by  $\mathcal{L}$ . Elements of  $\mathcal{L}$  are equivalence classes of Cauchy sequences of continuous functions on  $[0, 1]$ ; it is not clear yet whether they can be identified with actual functions on  $[0, 1]$ . By the end of this section, we will understand in what sense such an identification can be made. Until then, we'll denote the metric in the completion by  $d_1$  rather than  $\|\cdot\|_1$  to avoid confusion. We will say that a sequence  $(f_n)$  converges in  $\mathcal{L}$  to a limit  $f$  if  $d_1(f_n, f) \rightarrow 0$ .

We begin by observing that the (Riemann) integral, thought of as a continuous linear functional on  $C([0, 1])$ , can be extended to a continuous linear functional on  $\mathcal{L}$ . Let us write

$$I(f) = \|f\|_1 = \int_0^1 f(t) dt.$$

Observe that  $I$  is a 1-Lipschitz linear map from  $C([0, 1])$  to  $\mathbb{R}$ :

$$|I(f) - I(g)| \leq \int_0^1 |f(t) - g(t)| dt = d_1(f, g).$$

**Lemma 2.5.67.**  *$I$  can be extended to a 1-Lipschitz linear map from  $\mathcal{L}$  to  $\mathbb{R}$ . Furthermore,  $f \geq 0$  implies  $I(f) \geq 0$  for all  $f \in \mathcal{L}$ .*

In the preceding lemma, we gave an inequality of the form  $f \geq 0$  for an element  $f \in \mathcal{L}$ . The meaning of this inequality is not immediately clear. We say that  $f \geq 0$ ,  $f \in \mathcal{L}$ , if  $f$  is the limit in  $(C([0, 1]), \|\cdot\|_1)$  of a sequence of positive continuous functions. For  $f, g \in \mathcal{L}$ , we say that  $f \leq g$  if  $g - f \geq 0$ .

*Proof.* The existence of the extended map  $I : \mathcal{L} \rightarrow \mathbb{R}$  and its regularity (1-Lipschitz) are an easy application of Proposition 2.3.49. Linearity of the extension follows from linearity of the original map  $I : C([0, 1]) \rightarrow \mathbb{R}$  by passing to the limit. A similar argument establishes the *coercivity* estimate:  $f \geq 0$  implies  $I(f) \geq 0$  for all  $f \in \mathcal{L}$ .  $\square$

Elements of  $\mathcal{L}$  are equivalence classes  $[(f_n)]$  of Cauchy sequences of continuous functions on  $[0, 1]$  (relative to  $\|\cdot\|_1$ ). By the lemma, to each such equivalence class  $f$  there corresponds a number (the “integral” of  $f$ ); this correspondence is linear, monotone and 1-Lipschitz from  $(\mathcal{L}, \|\cdot\|_1)$  to  $\mathbb{R}$ . This extension of the Riemann integral is called the *Daniell integral*. A different, but equivalent, approach to completing the Riemann integral, due to Lebesgue, consists in first defining a notion of measure, then proceeding via the direct construction of Lebesgue measure, open sets, and then (Lebesgue) measurable and integrable functions. We will say more about the Lebesgue theory at the end of this section.

**Theorem 2.5.68 (Monotone Convergence Theorem).** *Let  $(f_n)$  be an increasing sequence of elements of  $\mathcal{L}$  so that  $\sup_n I(f_n) < \infty$ . Then there exists  $f \in \mathcal{L}$  so that  $f_n$  converges in  $\mathcal{L}$  to  $f$  and  $I(f) = \lim_n I(f_n)$ .*

Notice that all of the elements  $f_n$ ,  $n = 1, 2, 3, \dots$ , and  $f$  are elements of  $\mathcal{L}$ , hence equivalence classes of Cauchy sequences in  $X$ .

*Proof.* By monotonicity,  $(I(f_n))$  is an increasing sequence of real numbers which is bounded above. Hence  $L = \lim_n I(f_n)$  exists. For all  $\epsilon > 0$  there exists  $N$  so that

$$L - \epsilon < I(f_N) \leq I(f_n) \leq L$$

for all  $n \geq N$ . It follows that  $(f_n)$  is a Cauchy sequence in  $(\mathcal{L}, d_1)$ :  $n, m \geq N$  implies

$$d_1(f_n, f_m) = I(|f_n - f_m|) = I(f_n) - I(f_m) < \epsilon.$$

Completeness of  $(\mathcal{L}, d_1)$  yields the limit element  $f \in \mathcal{L}$ . The final claim comes from the continuity of the integral, see Lemma 2.5.67.  $\square$

Now we're prepared to show how the elements of  $\mathcal{L}$  can be viewed as functions. To do this, it's necessary to pass to another quotient space, i.e., to impose an equivalence relation on the space of functions from  $[0, 1]$  to  $\mathbb{R}$ . To get a handle on this equivalence relation, let's begin with the following natural question: if  $(f_n)$  is a sequence in  $C([0, 1])$  converging in  $\mathcal{L}$  to the zero function  $f_\infty(x) \equiv 0$ , how large can the set of points  $x \in [0, 1]$  be on which  $f_n(x)$  does **not** converge to 0?

**Example 2.5.69.** The sequence  $f_n(x) = x^n$  shows that convergence of  $(f_n(x))$  to zero can fail at a single point ( $x = 1$ ).

**Exercise 2.5.70.** Show that convergence of  $(f_n(x))$  to zero can fail at a countable set of points.

The length  $|U|$  of an open interval  $U = (a, b) \subset \mathbb{R}$  is equal to  $b - a$ .

**Definition 2.5.71.** A set  $A \subset [0, 1]$  is said to have *measure zero* if for all  $\epsilon > 0$  there exists a collection of open intervals  $U_i$ ,  $i \in \mathbb{N}$ , so that  $A \subset \cup_i U_i$  and  $\sum_i |U_i| < \epsilon$ .

**Examples 2.5.72.** 1. The rationals  $\mathbb{Q} \cap [0, 1]$  have measure zero. In fact, for any  $\epsilon > 0$  the sequence of open intervals  $U_i = (q_i - 2^{-i-1}\epsilon, q_i + 2^{-i-1}\epsilon)$  covers  $\mathbb{Q} \cap [0, 1] = \{q_1, q_2, \dots\}$  and has  $\sum_i |U_i| = \epsilon/2$ . (The same argument shows that any countable subset of  $[0, 1]$  has measure zero.)

2.  $[0, 1]$  does not have measure zero. In fact, if  $\{U_i\}$  is any collection of open intervals with  $[0, 1] \subset \cup_i U_i$ , then  $\sum_i |U_i| \geq 1$ . To prove this, we appeal to the Heine-Borel theorem, which we will state and prove in the following section. It implies that  $[0, 1]$  is compact: from every collection of open intervals covering  $[0, 1]$  we can select a finite subcollection which still covers  $[0, 1]$ . Choose such a subcollection  $U_{i_1}, \dots, U_{i_k}$ . Then

$$\sum_i |U_i| \geq \sum_{j=1}^k |U_{i_j}| \geq 1$$

by an easy induction using the triangle inequality.

3. The standard  $\frac{1}{3}$  Cantor set  $C \subset [0, 1]$  has measure zero. Recall that

$$C = [0, 1] \setminus \bigcup_{m=0}^{\infty} \bigcup_{i \in I_m} O_{mi}, \quad O_{mi} = \left( \frac{3i+1}{3^{m+1}}, \frac{3i+2}{3^{m+1}} \right),$$

where  $I_m$  consists of those integers  $i$  with  $0 \leq i < 3^m$  whose ternary expansion involves only the digits 0 and 2. (For example,  $I_2 = \{0, 2, 6, 8\}$ .) Observe that with this definition for  $I_m$ , all of the open intervals  $O_{mi}$ ,  $m \geq 0$ ,  $i \in I_m$  are disjoint. Alternatively,  $C$  can be described as the set of all real numbers in  $[0, 1]$  which have a ternary expansion involving only the digits 0 and 2.  $C$  is an uncountable subset of  $[0, 1]$  of zero length, in the sense that

$$\sum_m \sum_{i \in I_m} |O_{mi}| = 1.$$

To see that  $C$  has measure zero, it's helpful to rewrite  $C$  as a decreasing intersection of closed sets:

$$C = \bigcap_{m=0}^{\infty} \bigcup_{i \in I_m} A_{mi}, \quad A_{mi} = \left[ \frac{3i}{3^m}, \frac{3i+1}{3^m} \right],$$

where  $I_m$  is as before.

Suppose that  $0 < \epsilon < 1$ , and choose  $m$  so that

$$\left( \frac{2}{3} \right)^m < \frac{1}{3} \epsilon.$$

For each  $i \in I_m$ , let

$$U_{mi} = \left( \frac{3i - \epsilon}{3^m}, \frac{3i + 1 + \epsilon}{3^m} \right) \supset A_{mi}.$$

Then  $C \subset \bigcup_{i \in I_m} U_{mi}$  and

$$\sum_m \sum_{i \in I_m} |U_{mi}| = (\#I_m) \left( \frac{1 + 2\epsilon}{3^m} \right) = (1 + 2\epsilon) \left( \frac{2}{3} \right)^m < \epsilon.$$

Since  $\epsilon$  was arbitrary,  $C$  has measure zero.

4. The measure zero property is *hereditary*: if  $A$  has measure zero and  $B \subset A$ , then  $B$  has measure zero.

**Proposition 2.5.73.** *Let  $A \subset [0, 1]$  be a closed set. Then  $A$  has measure zero if and only if there exists a sequence  $(f_n)$  in  $C([0, 1])$  converging in  $\mathcal{L}$  to the zero function so that  $A = \{x : f_n(x) \text{ does not converge to } 0\}$ .*

A property  $P$  of real numbers is said to hold *almost everywhere (a.e.)* if  $\{x \in [0, 1] : P \text{ does not hold at } x\}$  has measure zero. With this terminology, the reverse implication in Proposition 2.5.73 can be stated as follows: convergence in  $\mathcal{L}$  implies pointwise convergence almost everywhere.

**Exercise 2.5.74.** Prove Proposition 2.5.73.

As a corollary of the preceding theorem, we see that elements of  $\mathcal{L}$  can be identified with functions on  $[0, 1]$ , up to modification on sets of measure zero.

**Theorem 2.5.75.** *Let  $f \in \mathcal{L}$ . Then there exists a sequence of functions  $(f_n) \subset C([0, 1])$  converging to  $f$  in  $\mathcal{L}$  which converge pointwise almost everywhere to a function which we denote by  $f(x)$ . If  $(g_n)$  is another sequence converging to  $f$  in  $\mathcal{L}$  and pointwise to  $g(x)$  almost everywhere, then  $g(x) = f(x)$  a.e.*

*Proof.* Choose a sequence of functions  $(f_n)$  in  $C([0, 1])$  so that  $d_1(f_n, f) < 4^{-n}$  for all  $n$ . For each  $n$  let

$$O_n = \{x \in [0, 1] : |f_n(x) - f_{n+1}(x)| > 2^{-n}\}.$$

This is the pullback of the open set  $(2^{-n}, \infty)$  by the continuous function  $|f_n - f_{n+1}|$ , hence is open. We may then write  $O_n = \bigcup_m U_{nm}$  for some disjoint open intervals  $U_{nm} = (a_{nm}, b_{nm})$ . Observe that

$$\begin{aligned} \sum_m |U_{nm}| &\leq \sum_m \int_{a_{nm}}^{b_{nm}} 2^n |f_n(x) - f_{n+1}(x)| dx \\ &\leq 2^n \|f_n - f_{n+1}\|_1 \\ &\leq 2^n (d_1(f_n, f) + d_1(f_{n+1}, f)) < 2^{1-n}. \end{aligned}$$

Let  $A = \bigcap_{n \geq 1} \bigcup_{k \geq n} O_k$ . For each  $n$ ,  $A$  can be covered by the open intervals comprising  $\bigcup_{k \geq n} O_k$ , which have total length at most  $\sum_{k \geq n} 2^{1-k} = 2^{2-n}$ . Thus  $A$  has measure zero.

If  $x \in [0, 1] \setminus A$ , then there exists  $N \geq 1$  so that  $x \notin \bigcup_{n \geq N} O_n$ , i.e.,  $x \in \bigcap_{n \geq N} [0, 1] \setminus O_n$ . Thus  $|f_n(x) - f_{n+1}(x)| \leq 2^{-n}$  for all  $n \geq N$ , so

$$|f_n(x) - f_m(x)| \leq \sum_{k=n}^{m-1} |f_k(x) - f_{k+1}(x)| \leq \sum_{k=n}^{m-1} 2^{-k} < 2^{1-n} \leq 2^{1-N}$$

for all  $m > n \geq N$ . Thus  $(f_n(x))$  is a Cauchy sequence in  $\mathbb{R}$ , and so converges to a value which we denote  $f(x)$ . In other words,  $(f_n)$  converges pointwise almost everywhere to a function  $f(x)$ .

Now suppose that  $(g_n)$  is another sequence in  $C([0, 1])$  converging to  $f$  in  $\mathcal{L}$  and converging pointwise almost everywhere to a function  $g(x)$ . Passing to a subsequence, which we continue to denote by  $(g_n)$ , we may assume that  $d_1(g_n, f) < 4^{-n}$  for all  $n$ . Let

$$V_n = \{x \in [0, 1] : |g_n(x) - f_n(x)| > 2^{-n}\}.$$

Again,  $V_n$  is open since  $|g_n - f_n|$  is continuous. Arguing as before, we find that  $B = \bigcap_{n \geq 1} \bigcup_{k \geq n} V_k$  has measure zero. On the complement of  $B$  we have  $|g_n(x) - f_n(x)| \rightarrow 0$ . Since  $f_n(x) \rightarrow f(x)$  for all  $x \in [0, 1] \setminus A$ ,  $f(x) = g(x)$  for  $x \in [0, 1] \setminus (A \cup B)$ . Thus  $g = f$  a.e.  $\square$

The preceding theorem shows that elements of  $\mathcal{L}$  can be identified with equivalence classes of functions, where the equivalence relation is equality almost everywhere. The resulting quotient space  $L = \mathcal{L} / \sim$  is called the space of (*Lebesgue*)

*integrable functions.* The linear functional  $I : \mathcal{L} \rightarrow \mathbb{R}$  which extends the Riemann integral on  $C([0, 1])$  descends to a map on  $L$  which we continue to denote by  $I$ . The quantity  $I(f)$  is called the *Lebesgue integral* of  $f$  over  $[0, 1]$ .

**Remark 2.5.76.** The approach to the Lebesgue integral developed in this section (beginning with the space  $L$  of integrable functions) is somewhat unorthodox. The more traditional development of the subject begins with the notion of *Lebesgue measure*  $\mu$ , which extends the usual length measure for intervals to a wider class of subsets of  $[0, 1]$ . Then a notion of integration with respect to  $\mu$  can be defined, giving a linear functional  $f \mapsto \int_0^1 f d\mu$  on a certain completion of  $C([0, 1])$ . The Lebesgue space  $L$  consists of all functions  $f$  for which  $\int_0^1 f d\mu$  exists and is finite.

**Exercise 2.5.77.** Construct the Lebesgue measure  $\mu$  from the space  $L$  of Lebesgue integrable functions  $f$  with integral  $I(f)$  as defined in this section. More precisely, let  $\Sigma$  be the collection of all subsets  $E \subset [0, 1]$  whose characteristic function  $\chi_E$  belongs to  $L$  (i.e.,  $\chi_E$  agrees a.e. with some function  $g \in L$ ). Then define  $\mu(E) = I(\chi_E)$ . Show that

- $\Sigma$  is a  $\sigma$ -algebra:  $E, F \in \Sigma \Rightarrow E \cap F, E \cup F, E \setminus F \in \Sigma$ ,
- $\mu(E \cap F) + \mu(E \cup F) = \mu(E) + \mu(F)$  for all  $E, F \in \Sigma$ ,
- $\mu(E) = 0$  if and only if  $E$  has measure zero,
- $\mu(\bigcup_m E_m) = \sum_m \mu(E_m)$  if  $(E_m)$  is a collection of disjoint sets in  $\Sigma$ ,
- $\Sigma$  contains all open intervals, and  $\mu(U) = d - c$  if  $U = (c, d)$ .

One final remark is worth mentioning. In the approach taken here, the integral  $I(f)$ ,  $f \in L$ , is defined abstractly; it is not clear how to recover its value from the actual values taken on by (a representative of)  $f$ . In the alternate approach to the Lebesgue integral discussed in Remark 2.5.76, the value of the integral is determined explicitly in terms of the actual values taken on by the function.

## 2.6 Compactness

Compactness is arguably the single most important concept in analysis. It is the essential feature which guarantees the existence of limits in a wide variety of cases. There are several definitions of compactness which are all equivalent in the setting of metric spaces. Some of these definitions can be extended to the more general class of topological spaces (see section 2.8) where they give different notions.

**Definition 2.6.78.** A collection  $\{U_i\}_{i \in I}$  of subsets of a set  $X$  is a *cover* of  $X$  if  $X = \bigcup_{i \in I} U_i$ . A *subcover* of a cover  $\{U_i\}_{i \in I}$  is a subcollection  $\{U_i\}_{i \in I'}$ ,  $I' \subset I$ , for which  $X = \bigcup_{i \in I'} U_i$ .

A cover  $\{U_i\}$  of a metric space  $(X, d)$  is an *open cover* if the sets  $U_i$  are open sets.

**Definition 2.6.79.** A metric space  $(X, d)$  is *compact* if every open cover admits a subcover of finite cardinality.

To illustrate the strength of this notion, we next present a number of properties of compact spaces.

**Definition 2.6.80.** A metric space  $(X, d)$  is *bounded* if  $X = B(x_0, r)$  for some  $x_0 \in X$  and  $r > 0$ . It is *totally bounded* if for every  $\epsilon > 0$  there exists a finite collection of open balls of radius  $\epsilon$  covering  $X$ . (The centers of such a collection of balls are called an  $\epsilon$ -net in  $(X, d)$ .)

A sequence  $(x_n)$  *subconverges* to  $x_\infty$  if there exists a subsequence  $(x_{n_k})$  which converges to  $x_\infty$ ;  $(x_n)$  is *subconvergent* if it has a convergent subsequence.  $(X, d)$  is *sequentially compact* if every sequence is subconvergent.

$(X, d)$  has the *finite intersection property* if every collection  $\{F_i\}_{i \in I}$  of closed sets in  $X$ , with the property that every finite subcollection has nonempty intersection, necessarily has nonempty intersection:

$$\forall I' \subset I, I' \text{ finite } \bigcap_{i \in I'} F_i \neq \emptyset \quad \Rightarrow \quad \bigcap_{i \in I} F_i \neq \emptyset.$$

**Proposition 2.6.81.** *All compact metric spaces are*

- (a) *bounded,*
- (b) *totally bounded,*
- (c) *separable,*
- (d) *sequentially compact,*
- (e) *complete.*

Moreover, they

- (f) *have the finite intersection property.*

Finally,

- (g) closed subsets of compact spaces are compact,
- (h) continuous images of compact spaces are compact,
- (i) a continuous real-valued function defined on a compact metric space achieves its maximum and minimum, and
- (j) continuous maps from compact spaces are uniformly continuous.

*Proof.* (a) Fix  $x_0 \in X$  and consider the sequence of open sets  $U_n = B(x_0, n)$ .

(b) Consider the collection of open sets  $U_x = B(x, \epsilon)$ ,  $x \in X$ .

(c) Apply the total boundedness condition for  $\epsilon = \frac{1}{k}$ ,  $k \in \mathbb{N}$ , to construct finite  $\frac{1}{k}$ -nets  $N_k$  for each  $k$ . Then  $\bigcup_{k=1}^{\infty} N_k$  is a countable dense set.

(d) Suppose that  $(x_n)$  is a non-subconvergent sequence in  $(X, d)$ . Then for each  $x \in X$ , there exists  $\epsilon_x > 0$  so that  $B(x, \epsilon_x)$  contains only finitely many terms in the sequence. By compactness, there is a finite subcover  $\{B(x_1, \epsilon_{x_1}), \dots, B(x_N, \epsilon_{x_N})\}$  of the cover  $\{B(x, \epsilon_x)\}_{x \in X}$ . This subcover can contain at most finitely many terms from the sequence. It follows that the sequence  $(x_n)$  has finite range; only finitely many values are taken on by the elements of the sequence. Then  $(x_n)$  has a constant (hence convergent) subsequence.

(e) Let  $(x_n)$  be a Cauchy sequence. By sequential compactness,  $(x_n)$  subconverges to  $x_\infty$  in  $X$ . But a Cauchy sequence which subconverges to  $x_\infty$ , necessarily converges to  $x_\infty$ . Thus  $(X, d)$  is complete.

(f) Take complements!

(g) Let  $A$  be a closed subset of  $X$ , and let  $(x_n)$  be a sequence in  $A$ . Then  $(x_n)$  subconverges to  $x_\infty$  in  $X$ . Since  $A$  is closed,  $x_\infty \in A$ . Thus  $(x_n)$  subconverges to  $x_\infty$  in  $A$ , so  $A$  is compact.

(h) Let  $f : (X, d) \rightarrow (Y, d')$  be continuous, and let  $\{V_i\}_{i \in I}$  be an open cover of  $Y$ . Then  $\{U_i\}_{i \in I}$ ,  $U_i = f^{-1}(V_i)$ , is an open cover of  $X$ , so there exists a finite subcover  $\{U_{i_1}, \dots, U_{i_N}\}$ . Then  $\{V_{i_1}, \dots, V_{i_N}\}$  is a finite subcover of  $Y$ .

(i) Let  $f : (X, d) \rightarrow \mathbb{R}$  be continuous, with  $(X, d)$  compact. By (h),  $f(X) \subset \mathbb{R}$  is compact. By the Heine–Borel theorem 2.6.84,  $f(X)$  is closed and bounded. Thus  $\sup_{x \in X} f(x)$  and  $\inf_{x \in X} f(x)$  are achieved, i.e., there exist points  $x_\pm \in X$  so that  $f(x_+) = \max_{x \in X} f(x)$  and  $f(x_-) = \min_{x \in X} f(x)$ .

(j) Let  $\epsilon > 0$ . For each  $x \in X$  there exists  $\delta_x > 0$  so that  $f(B(x, 2\delta_x)) \subset B(fx, \epsilon/2)$ . The sets  $U_x = B(x, \delta_x)$  form an open cover of  $X$ ; choose a finite subcover  $U_{x_1}, \dots, U_{x_N}$ . Let  $\delta = \min\{\delta_{x_1}, \dots, \delta_{x_N}\}$ . If  $d(x, y) < \delta$ , then there exists  $i$ ,  $1 \leq i \leq N$ , so that  $x, y \in B(x_i, 2\delta_{x_i})$ . Then  $fx, fy \in B(fx_i, \epsilon/2)$  so  $d(fx, fy) < \epsilon$ .  $\square$

The following theorem gives some of the equivalent notions of compactness in metric spaces.

**Theorem 2.6.82.** *Let  $(X, d)$  be a metric space. The following are equivalent:*

- (i)  $(X, d)$  is compact,
- (ii)  $(X, d)$  has the finite intersection property for countable families  $\{F_i\}_{i=1}^{\infty}$  of closed sets,

(iii)  $(X, d)$  is sequentially compact,

(iv)  $(X, d)$  is complete and totally bounded.

*Proof.* We have already shown that (i) implies (ii), (iii) and (iv). To complete the chain of implications, we will prove that (ii)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (i).

(ii)  $\Rightarrow$  (iii): Assume that  $(X, d)$  has the finite intersection property for countable collections  $\{F_i\}_{i=1}^{\infty}$  of closed sets. We make use of the following elementary lemma, whose proof we leave to the reader.

**Lemma 2.6.83.** *Let  $(x_n)$  be a sequence of distinct elements of  $(X, d)$ , and let  $A = \{x_1, x_2, \dots\}$  be the range of this sequence. Then every element of  $\bar{A} \setminus A$  arises as the limit of a subsequence of  $(x_n)$ .*

Returning to the proof, suppose that  $(X, d)$  is not sequentially compact. Then there exists a sequence  $(x_n)$  with no convergent subsequence. We may assume that the elements of  $(x_n)$  are all distinct. By the lemma, the sets  $F_N = \{x_N, x_{N+1}, \dots\}$ ,  $N \geq 1$ , are all closed. Moreover, finite intersections of these sets are nonempty. But  $\bigcap_N F_N = \emptyset$ . This contradicts the assumption.

(iii)  $\Rightarrow$  (iv): First, suppose that  $(x_n)$  is a Cauchy sequence in  $(X, d)$ . By assumption,  $(x_n)$  subconverges to some  $x_{\infty}$ . But then  $(x_n)$  converges to  $x_{\infty}$ . Thus  $(X, d)$  is complete.

Suppose that  $(X, d)$  is not totally bounded. Then there exists an infinite  $\epsilon$ -net in  $(X, d)$  for some  $\epsilon > 0$ . Extract from this net a sequence of points; this sequence can have no convergent subsequence.

(iv)  $\Rightarrow$  (i): Suppose that  $\mathcal{U} := \{U_i\}_{i \in I}$  is an open cover of  $(X, d)$  with no finite subcover. Define a sequence of **closed** balls  $A_n = \bar{B}(x_n, r_n) = \{y \in X : d(y, x_n) \leq r_n\}$  in  $X$  as follows: Let  $x_0 \in X$  be arbitrary and choose  $R$  so large that  $A_0 = \bar{B}(x_0, R) = X$ . By assumption,  $A_0$  has no finite subcover from  $\mathcal{U}$ . Next, choose  $A_1 = \bar{B}(x_1, R/2)$  having no finite subcover from  $\mathcal{U}$ . This can be done since  $(X, d)$  is totally bounded; cover  $X$  with a finite collection of balls of radius  $R/2$  and observe that  $A_1$  may be chosen to be one of the balls. Then choose  $A_2 = \bar{B}(x_2, R/4)$  having no subcover from  $\mathcal{U}$  and with  $A_2 \cap A_1 \neq \emptyset$ . Indeed, since  $(A_1, d)$  is totally bounded we may cover  $A_1$  with a finite collection of balls of radius  $R/4$  and choose one of these balls to be  $A_2$ . Continuing in this fashion, we define closed balls  $A_n$  as above satisfying  $A_n \cap A_{n-1} \neq \emptyset$  for all  $n$ .

We claim that the sequence  $(x_n)$  is Cauchy. Indeed, if  $n \geq m \geq N$ , then

$$d(x_n, x_m) \leq \sum_{m < i \leq n} d(x_i, x_{i-1}) \leq \sum_{m < i \leq n} \frac{R}{2^i} + \frac{R}{2^{i-1}} \leq C \frac{R}{2^N}$$

for a suitable constant  $C$ . This tends to zero as  $N \rightarrow \infty$ , so  $(x_n)$  is Cauchy. Since  $(X, d)$  is complete,  $(x_n)$  converges to some limit point  $x_{\infty}$ . Since  $\mathcal{U}$  is an open cover of  $X$ ,  $x_{\infty} \in U_{i_0}$  for some  $i_0$ . For sufficiently large  $N$ ,  $A_N \subset U_{i_0}$ , contradicting the fact that  $A_N$  has no finite subcover from  $\mathcal{U}$ .  $\square$

We finish with two additional facts about compact sets.

**Theorem 2.6.84 (Heine–Borel theorem).** *A set in  $\mathbb{R}^n$  is compact if and only if it is closed and bounded.*

*Proof.* It suffices to prove the reverse implication. If  $A \subset \mathbb{R}^n$  is closed and bounded, then it is complete (as a closed subset of a complete space), and totally bounded (exercise). By Theorem 2.6.82,  $A$  is compact.  $\square$

The *diameter* of a set  $A \subset X$  is

$$\text{diam } A = \sup_{x,y \in A} d(x,y).$$

A *Lebesgue number* for an open cover  $\{U_i\}$  of a metric space  $(X, d)$  is a positive number  $\delta$  so that every set  $A$  of diameter less than  $\delta$  is contained in one of the sets  $U_i$ .

**Theorem 2.6.85.** *Every open cover of a compact metric space admits a Lebesgue number.*

*Proof.* Suppose that  $\{U_i\}$  is an open cover with no Lebesgue number. Then there exist nonempty sets  $A_n$ ,  $\text{diam } A_n \rightarrow 0$ , so that  $A_n \not\subset U_i$  for any  $i$ . Define a sequence  $(x_n)$  by choosing  $x_n \in A_n$  for each  $n$ . By hypothesis,  $(x_n)$  subconverges to  $x_\infty$ . Since  $\{U_i\}$  is an open cover,  $x_\infty \in U_{i_0}$  for some  $i_0$ . Pick  $r > 0$  so that  $B(x_\infty, 2r) \subset U_{i_0}$ . If  $n$  is chosen sufficiently large, then  $d(x_n, x_\infty) < r$  and  $\text{diam } A_n < r$ ; this ensures that  $A_n \subset B(x_\infty, 2r) \subset U_{i_0}$ . This is a contradiction.  $\square$

## 2.7 Fixed point theorems and applications to differential equations

Let  $(X, d)$  and  $(Y, d')$  be metric spaces. A *contractive* map  $F : (X, d) \rightarrow (Y, d')$  is a map which is  $k$ -Lipschitz for some  $k < 1$ . A *fixed point* for a map  $T : X \rightarrow X$  ( $X$  any set) is a point  $x \in X$  so that  $T(x) = x$ .

**Theorem 2.7.86 (Contraction Mapping Principle).** *Let  $(X, d)$  be a complete metric space and let  $T : (X, d) \rightarrow (X, d)$  be a contractive map. Then  $T$  has a unique fixed point.*

*Proof.* Let  $x_0$  be an arbitrary point in  $X$  and consider the sequence  $(x_n)$  defined by the recursion  $x_{n+1} = T(x_n)$ . Note that

$$d(x_{n+1}, x_n) = d(T(x_n), T(x_{n-1})) \leq kd(x_n, x_{n-1}) \leq \cdots \leq k^n d(x_1, x_0)$$

so

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq (k^n + \cdots + k^{m-1})d(x_1, x_0) \leq k^n \frac{d(x_1, x_0)}{1-k}$$

if  $m \geq n$ . In particular,  $(x_n)$  is Cauchy and hence converges to a limit  $x_\infty$  by the completeness of  $(X, d)$ . The continuity of the distance function implies that

$$d(T(x_\infty), x_\infty) = d(\lim_n T(x_n), \lim_n x_{n+1}) = \lim_n d(T(x_n), x_{n+1}) = 0$$

so  $x_\infty$  is a fixed point for  $T$ . If  $y_\infty$  is another fixed point for  $T$ , then

$$d(x_\infty, y_\infty) = d(T(x_\infty), T(y_\infty)) \leq kd(x_\infty, y_\infty)$$

which implies that  $y_\infty = x_\infty$ . □

**Exercise 2.7.87.** Find a contractive map of a noncomplete metric space which fails to have any fixed points. Find a 1-Lipschitz map of a complete metric space which fails to have any fixed points.

**Exercise 2.7.88.** Let  $(X, d)$  be a compact metric space and let  $T : (X, d) \rightarrow (X, d)$  satisfy  $d(T(x), T(y)) < d(x, y)$  for all  $x, y \in X$ ,  $x \neq y$ . Show that  $T$  has a unique fixed point.

*Sketch of proof of Exercise 2.7.88.* Let  $x_0$  and  $(x_n)$  be as in the preceding proof. Choose  $x_\infty$  as the limit of a subsequence of the sequence  $(x_n)$ . Suppose that  $x_\infty \neq T(x_\infty)$ . Then we may choose  $\epsilon > 0$  and  $k < 1$  so that  $d(T(x), T(y)) \leq kd(x, y)$  for all  $x \in U := B(x_\infty, \epsilon)$  and all  $y \in V := B(T(x_\infty), \epsilon)$ . There exists  $N$  so that  $x_n \in U$  and  $T(x_n) \in V$  for all  $n \geq N$ . Then

$$d(x_n, T(x_n)) \leq k^{n-N} d(x_N, T(x_N)) \rightarrow 0$$

as  $n \rightarrow \infty$ , which contradicts the assumption that  $x_\infty \neq T(x_\infty)$ . □

We apply the Contraction Mapping Principle to a suitable function space to deduce the basic Existence and Uniqueness Theorem for (systems of) first order ODE's.

**Theorem 2.7.89 (Picard Existence and Uniqueness Theorem for ODE's).**  
 Let  $F : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a continuous function which is Lipschitz in the second variable:  $\exists L < \infty$  so that

$$|F(t, x) - F(t, y)| \leq L|x - y|$$

for all  $x, y \in \mathbb{R}^n$  and  $t \in \mathbb{R}$ . Let  $x_0 \in \mathbb{R}^n$  and  $t_0 \in \mathbb{R}$  and assume that there exist  $C_1, C_2 > 0$  so that

$$|F(t - t_0, x_0)| \leq C_1 e^{C_2|t|}$$

for all  $t \in \mathbb{R}$ . Then there exists a unique differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  so that

$$f'(t) = F(t, f(t)), \quad f(t_0) = x_0.$$

*Proof.* We may assume that  $t_0 = 0$ . Let  $\alpha > \max\{L, C_2\}$ , and let

$$X = \{f \in C(\mathbb{R} : \mathbb{R}^n) : \sup_{t \in \mathbb{R}} e^{-\alpha|t|} |f(t) - x_0| < \infty\}.$$

Here  $C(\mathbb{R} : \mathbb{R}^n)$  denotes the space of all continuous functions on  $\mathbb{R}$  taking values in  $\mathbb{R}^n$ . We equip  $X$  with the metric

$$d(f, g) = \sup_{t \in \mathbb{R}} e^{-\alpha|t|} |f(t) - g(t)|.$$

Then  $(X, d)$  is a complete metric space (the proof of this is similar to the proof in Chapter 2 that the space  $C_b(X)$  equipped with the metric  $d_\infty$  is complete). Define a map  $T \rightarrow C(\mathbb{R} : \mathbb{R}^n)$  by

$$T(f)(t) = x_0 + \int_0^t F(s, f(s)) ds. \quad (2.7.90)$$

Note that if  $f$  is continuous, then  $s \mapsto F(s, f(s))$  is continuous, so  $T(f)$  is continuous. We will prove that (i)  $T(f) \in X$ , and (ii)  $T : (X, d) \rightarrow (X, d)$  is a contraction mapping.

We begin with (ii). Suppose that  $f, g \in X$ . Then

$$\begin{aligned} |T(f)(t) - T(g)(t)| &= \left| \int_0^t F(s, f(s)) - F(s, g(s)) ds \right| \\ &\leq \int_0^{|t|} |F(s, f(s)) - F(s, g(s))| ds \\ &\leq L \int_0^{|t|} |f(s) - g(s)| ds \\ &\leq Ld(f, g) \int_0^{|t|} e^{\alpha s} ds = \frac{L}{\alpha}(e^{\alpha|t|} - 1)d(f, g) \end{aligned}$$

so

$$d(T(f), T(g)) \leq \frac{L}{\alpha}(1 - e^{-\alpha|t|})d(f, g).$$

Thus  $T : X \rightarrow C(\mathbb{R} : \mathbb{R}^n)$  is  $k$ -Lipschitz with  $k = \frac{L}{\alpha} < 1$ . To see that  $T : X \rightarrow X$ , we consider the constant function  $f_0(t) = x_0$  and estimate

$$\begin{aligned} d(T(f_0), f_0) &= \sup_{t \in \mathbb{R}} e^{-\alpha|t|} \left| \int_0^t F(s, x_0) ds \right| \\ &\leq \sup_{t \in \mathbb{R}} e^{-\alpha|t|} C_1 \int_0^{|t|} e^{C_2 s} ds \\ &= \frac{C_1}{C_2} \sup_t e^{-\alpha|t|} (e^{C_2|t|} - 1) \leq \frac{C_1}{C_2} < \infty \end{aligned}$$

so

$$d(T(f), f_0) \leq d(T(f), T(f_0)) + d(T(f_0), f_0) \leq kd(f, f_0) + d(T(f_0), f_0) < \infty$$

and  $T : X \rightarrow X$ .

By the Contraction Mapping Principle, there exists a unique function  $f \in X$  satisfying  $T(f) = f$ , i.e.,

$$f(t) = x_0 + \int_0^t F(s, f(s)) ds.$$

Then  $f(0) = x_0$  and an application of the Fundamental Theorem of Calculus gives

$$f'(t) = F(t, f(t)).$$

□

**Remark 2.7.91.** The preceding proof can be converted into an algorithm for computing the solution  $x = f(t)$ . Set  $f_0(t) = x_0$  and define the sequence of functions  $f_n = T^{(n)}(f_0)$ , i.e.,

$$f_{n+1}(t) = x_0 + \int_0^t F(s, f_n(s)) ds.$$

The functions  $f_n$  are called the *Picard iterates* for the differential equation

$$f'(t) = F(t, f(t)), \quad f(t_0) = x_0.$$

**Example 2.7.92.** Let  $n = 1$  and  $F(t, x) = x$ ,  $t_0 = 0$ ,  $x_0 = 1$ . The solution to the ODE  $x' = x$ ,  $x(0) = 1$ , is  $x(t) = e^t$ . The Picard iterates are  $f_0(t) = 1$ ,  $f_1(t) = 1 + t$ ,  $f_2(t) = 1 + t + \frac{1}{2}t^2$ ,  $\dots$ . Observe that these are the Taylor partial sums for the function  $e^t = \sum_{m=0}^{\infty} \frac{t^m}{m!}$ .

The function  $F(t, x) = x^2$  is **not** Lipschitz in the second variable (on all of  $\mathbb{R}$ ). The solution to the ODE  $x' = x^2$ ,  $x(0) = 1$ , can be obtained by separation of variables:

$$x(t) = \frac{1}{1-t}.$$

Notice that this solution is not continuous on all of  $\mathbb{R}$ ; there is a discontinuity at  $t = 1$ . We now state a local version of the Picard Existence and Uniqueness Theorem.

**Theorem 2.7.93 (Local Picard Theorem).** Let  $I \subset \mathbb{R}$  be a compact interval and let  $B \subset \mathbb{R}^n$  be a compact ball. Let  $I_0 \subset I$  and  $B_0 \subset B$  be the interiors of these sets (largest open subsets). Assume that  $F : I \times B \rightarrow \mathbb{R}^n$  is a continuous function which is Lipschitz in the second variable:  $\exists L < \infty$  so that

$$|F(t, x) - F(t, y)| \leq L|x - y|$$

for all  $x, y \in B$  and  $t \in I$ . Let  $(t_0, x_0) \in I_0 \times B_0$ . Then there exists  $h > 0$  and a unique differentiable function  $f : (t_0 - h, t_0 + h) \rightarrow \mathbb{R}^n$  so that

$$f'(t) = F(t, f(t)), \quad f(t_0) = x_0.$$

for all  $t$ ,  $|t - t_0| < h$ .

*Proof.* As before, we assume  $t_0 = 0$  and choose  $\alpha > L$ . Since  $I \times B$  is compact and  $F$  is continuous,  $M := \sup_{(t,x) \in I \times B} |F(t, x)| < \infty$ . Choose  $a$  and  $b$  so that  $(-a, a) \subset I_0$  and  $B(x_0, 2b) \subset B_0$ , and let

$$h = \min\left\{a, \frac{b}{M + Lb}\right\}.$$

Let

$$Y = \{f \in C([-h, h]) : f([-h, h]) \subset \overline{B(x_0, b)}\},$$

and equip  $Y$  with the maximum metric  $d_\infty$ . Define  $T : Y \rightarrow C(I)$  by the formula in (2.7.90). We claim that  $T : Y \rightarrow Y$ . Indeed, if  $f \in Y$  and  $|t| \leq h$ , then

$$\begin{aligned} |T(f)(t) - x_0| &= \left| \int_0^t F(s, f(s)) ds \right| \\ &\leq \int_0^{|t|} |F(s, f(s)) - F(s, x_0)| ds + \int_0^{|t|} |F(s, x_0)| ds \\ &\leq L \int_0^{|t|} |f(s) - x_0| ds + M|t|. \end{aligned}$$

Since  $f \in Y$ ,  $|f(t) - x_0| \leq b$  for all  $|t| \leq h$ . Thus

$$|T(f)(t) - x_0| \leq (Lb + M)h \leq b$$

and so  $T(f) \in Y$ .

Since  $Y$  is a closed subset of  $C(I)$ , it is complete. The same proof as before shows that  $T : Y \rightarrow Y$  is a contractive map. Thus there exists a unique fixed point  $f \in Y$  so that  $T(f) = f$ . From here the proof continues as in the proof of the global Picard theorem.  $\square$

**Example 2.7.94.** Let's return to our example:  $n = 1$  and  $F(t, x) = x^2$ ,  $t_0 = 0$ ,  $x_0 = 1$ . As mentioned before, the solution to the ODE  $x' = x^2$ ,  $x(0) = 1$ , is  $x(t) = 1/(1 - t)$ . The Picard iterates are  $f_0(t) = 1$ ,  $f_1(t) = 1 + t$ ,

$$f_2(t) = 1 + t + t^2 + \frac{1}{3}t^3,$$

and so on. Observe that the Taylor series for the solution is

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots$$

## 2.8 Topological spaces

We present only a few of the very basic notions of topology. The concept of a topology on a space axiomatizes some of the critical features of the collection of open sets in a metric space. There are many topologies on spaces which do not arise from metrics.

**Definition 2.8.95.** Let  $X$  be a set. A *topology* on  $X$  is given by a collection  $\mathcal{T}$  of subsets of  $X$ , called the *open sets* in  $X$ , so that (i)  $\emptyset, X \in \mathcal{T}$ , (ii)  $\mathcal{T}$  is closed under finite intersections ( $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$ ), and (iii)  $\mathcal{T}$  is closed under arbitrary unions ( $U_i \in \mathcal{T}, i \in I \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}$ ).

The pair  $(X, \mathcal{T})$  is called a *topological space*. A set  $F \subset X$  is called *closed* if  $X \setminus F$  is open. An open set  $U$  containing a point  $x_0 \in X$  is called an *open neighborhood* of  $x_0$ .

**Example 2.8.96.** Let  $(X, d)$  be any metric space, and let  $\mathcal{T}$  be the collection of all open sets in  $(X, d)$ . Then  $(X, \mathcal{T})$  is a topological space. The topology  $\mathcal{T}$  is called the topology *induced* by the metric  $d$ .

An example of a topology which does not arise from a metric is the following (when  $X$  is infinite).

**Example 2.8.97.** Let  $X$  be any set, and let  $\mathcal{T}$  be the collection of subsets of  $X$  consisting of the empty set together with all *cofinite sets* in  $X$ , i.e.,  $U \in \mathcal{T}$  if and only if  $U = \emptyset$  or  $X \setminus U$  is a finite set.

**Definition 2.8.98.** Let  $(X, \mathcal{T})$  be a topological space. A *base* for the topology is a subcollection  $\mathcal{B} \subset \mathcal{T}$  so that  $U \in \mathcal{T}$  if and only if  $U$  is the union of a collection of elements of  $\mathcal{B}$ .

**Proposition 2.8.99.** A collection  $\mathcal{B}$  is a base for a topology on  $X$  if and only if: (i)  $\mathcal{B}$  is a cover of  $X$ , and (ii) whenever  $B_1, B_2 \in \mathcal{B}$  with  $x \in B_1 \cap B_2$ , then there exists  $B_3 \in \mathcal{B}$  with  $x \in B_3 \subset B_1 \cap B_2$ .

*Proof.* The “only if” direction is clear. Suppose that  $\mathcal{B}$  is a collection of subsets satisfying (i) and (ii), and let  $\mathcal{T}$  be the collection of all unions of elements of  $\mathcal{B}$  (including the empty union). It is clear that  $\emptyset, X \in \mathcal{T}$ , and arbitrary unions of elements of  $\mathcal{T}$  are in  $\mathcal{T}$  by construction. Finally, if  $U, V \in \mathcal{T}$ , then write  $U = \bigcup_{B \in \mathcal{B}_1} B$ ,  $V = \bigcup_{C \in \mathcal{B}_2} C$  for some  $\mathcal{B}_1, \mathcal{B}_2 \subset \mathcal{B}$  and observe

$$U \cap V = \bigcup_{B \in \mathcal{B}_1} \bigcup_{C \in \mathcal{B}_2} (B \cap C).$$

By condition (ii),  $B \cap C$  is a union of elements of  $\mathcal{B}$  for each such  $B$  and  $C$ . Thus  $U \cap V \in \mathcal{T}$ .  $\square$

We transport to the setting of topological spaces many of the notions we previously considered for metric spaces:

**Continuity:** A map  $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{T}')$  between topological spaces is called *continuous* if  $U = f^{-1}(V) \in \mathcal{T}$  whenever  $V \in \mathcal{T}'$ . The notion of continuity in terms of sequences is no longer equivalent, at least in complete generality. The concepts of *nets* (or *Moore–Smith generalized sequences*) and *filters* were designed to deal with this issue; roughly speaking, these are extensions of the notion of sequence with the property that  $f$  is continuous if and only if it maps convergent nets (filters) to convergent nets (filters).

**Completeness:** This does **not** have a topological analog; it is a metric concept.

**Compactness:** A topological space  $(X, \mathcal{T})$  is *compact* if every open cover has a finite subcover. Again, the notion in terms of sequences (sequential compactness) is no longer equivalent.

Topological spaces also have certain new features which are not present, or not very interesting, in the metric space case. For example, they may satisfy one of a number of *separation conditions*. Roughly speaking, these conditions guarantee (to a greater or lesser extent) that the supply of open sets is sufficiently “rich”. A few such conditions follow. In all of these,  $(X, \mathcal{T})$  is a topological space.

$T_0$ : whenever  $x, y \in X$ ,  $x \neq y$ , then there exists  $U \in \mathcal{T}$  containing exactly one of the points  $x, y$ .

$T_1$ : whenever  $x, y \in X$ ,  $x \neq y$ , then there exists  $U \in \mathcal{T}$  containing  $x$  and not  $y$ , and there exists  $V \in \mathcal{T}$  containing  $y$  and not  $x$ .

$T_2$ : whenever  $x, y \in X$ ,  $x \neq y$ , then there exist disjoint sets  $U, V \in \mathcal{T}$  so that  $U$  contains  $x$ ,  $V$  contains  $y$ .

$T_3$ :  $T_1$  + whenever  $x \in X$  and  $B \subset X$  is closed, then there exist disjoint sets  $U, V \in \mathcal{T}$  so that  $U$  contains  $x$  and  $V \supset B$ .

$T_4$ :  $T_1$  + whenever  $A, B \subset X$  are disjoint closed sets, then there exist disjoint sets  $U, V \in \mathcal{T}$  so that  $U \supset A$  and  $V \supset B$ .

And this isn’t all! Some books define  $T_{3\frac{1}{2}}$  spaces ... Any space which is  $T_i$  for some  $i = 0, 1, 2, 3, 4$  is also  $T_j$  for all  $j \leq i$ .

$T_1$  spaces are interesting because this is the weakest separation condition which guarantees the (reasonable) condition that all sets consisting of one point are closed sets. (This also helps to explain why  $T_1$  is included as part of the definition of  $T_3$  and  $T_4$ ; otherwise it is not clear that  $T_i \Rightarrow T_2$ ,  $i = 3, 4$ .)

**Exercise 2.8.100.** Prove that  $(X, \mathcal{T})$  is  $T_1$  if and only if every singleton set is closed.

$T_2$  is the best-known of these separation conditions. It is also known as the *Hausdorff separation condition*. One of the reasons for its importance is that it is

the weakest of these conditions which holds for every metric space. Indeed, if  $x, y$  are two distinct points in a metric space  $(X, d)$ , then

$$U = B(x, \frac{1}{3}d(x, y)) \quad \text{and} \quad V = B(y, \frac{1}{3}d(x, y))$$

are disjoint open sets with  $x \in U$  and  $y \in V$ .

**Definition 2.8.101.** Let  $(X_i, \mathcal{T}_i)_{i=1}^{\infty}$  be a countable collection of topological spaces. The *product topology* (or *Tychonoff topology*)  $\mathcal{T}$  on the product space

$$\prod_{i=1}^{\infty} X_i = X_1 \times X_2 \times \cdots = \{x = (x_1, x_2, \dots) : x_i \in X_i \forall i\}$$

is the topology obtained from the basis  $\mathcal{B}$  consisting of all sets of the form  $U_1 \times U_2 \times \cdots$ , where  $U_i \in \mathcal{T}_i$  for all  $i$  and  $U_i = X_i$  for all but finitely many  $i$ .

We omit the general proof of the following theorem, which is however one of the most important theorems of point-set topology. It holds in much greater generality than given here. The cleanest proof uses the reformulation of topological notions in terms of nets or filters. There is also an interesting proof in terms of the finite intersection property; for a very readable account see section 5.1 of “*Topology: a first course*” by J. Munkres.

**Theorem 2.8.102 (Tychonoff product theorem).**  $(\prod_i X_i, \mathcal{T})$  is compact if and only if  $(X_i, \mathcal{T}_i)$  is compact for all  $i$ .

The case of finite products is somewhat easier. It relies on the following intuitively obvious lemma.

**Lemma 2.8.103 (Tube Lemma).** Let  $(X, \mathcal{T})$  and  $(Y, \mathcal{T}')$  be topological spaces with  $(Y, \mathcal{T}')$  compact. Let  $\mathcal{T} \times \mathcal{T}'$  be the product topology on  $X = X_1 \times X_2$ . If  $U \in \mathcal{T} \times \mathcal{T}'$  contains  $\{x\} \times Y$  then there exists  $V \in \mathcal{T}$  containing  $x$  so that  $V \times Y \subset U$ .

*Proof.* For each  $y \in Y$  we may choose sets  $V_y \in \mathcal{T}$  and  $W_y \in \mathcal{T}'$  so that  $(x, y) \in V_y \times W_y \subset U$ . The sets  $\{W_y\}_{y \in Y}$  form an open cover of  $Y$ ; since  $Y$  is compact, there exists a finite subcover  $W_{y_1}, \dots, W_{y_N}$ . Let  $V = V_{y_1} \cap \cdots \cap V_{y_N}$ . Then  $V \in \mathcal{T}$ ,  $x \in V$ , and

$$V \times Y = \bigcup_{j=1}^N V \times W_j \subset \bigcup_{j=1}^N V_j \times W_j \subset U.$$

□

*Proof of the Tychonoff product theorem 2.8.102.* We prove the result for the product of two factors; the general case follows by induction.

Assume that  $(X, \mathcal{T})$  and  $(Y, \mathcal{T}')$  are compact, let  $\mathcal{T} \times \mathcal{T}'$  be the product topology on  $X \times Y$ , and let  $\mathcal{U} \subset \mathcal{T} \times \mathcal{T}'$  be an open cover of  $X \times Y$ . For each  $x \in X$ , the slice  $\{x\} \times Y$  is homeomorphic with  $Y$ , hence compact. Thus there exists a finite number of elements  $U_1, \dots, U_M \in \mathcal{U}$  whose union contains  $\{x\} \times Y$ . By the Tube Lemma, there exists  $V_x \in \mathcal{T}$  so that  $\{x\} \times Y \subset V_x \times Y \subset U_1 \cup \cdots \cup U_M$ . The sets  $\{V_x\}_{x \in X}$  form an open cover of  $X$ ; by compactness of  $X$  there exists a finite subcover. Then  $X \times Y$  can be covered by finitely many tubes  $V_x \times Y$ , each of which can be covered by finitely many elements of  $\mathcal{U}$ . Thus  $X \times Y$  can be covered by finitely many elements of  $\mathcal{U}$ . □