Mathematics 595 CAP/TRA Homework #1
due Friday, 9/2

1: (a) Give an example of a vector space V and an element $T \in L(V, V)$ which is injective but not surjective.

(b) Give an example of a vector space V and an element $T \in L(V, V)$ which is surjective but not injective.

2: A linear map from a vector space V to the base field $\mathbb{F}$ is called a *linear functional* on V. The space $L(V, \mathbb{F})$ of linear functionals on V is called the *dual space* of V, and is denoted V^*.

Let V be a finite-dimensional vector space of dimension n, with basis $B = \{v_1, \ldots, v_n\}$.

(a) Show that every element $v \in V$ can be written uniquely in the form $v = \sum_{i=1}^n \lambda_i v_i$ for some $\lambda_i \in \mathbb{F}$.

(b) Show that the functions $k_i : V \to \mathbb{F}$ given by $k_i(v) = \lambda_i$, where λ_i is as in part (a), are linear functionals on V.

(c) Show that every linear functional on V can be written as a linear combination of the functionals $k_1, \ldots, k_n$. Deduce that V and V^* are isomorphic.

3: Let $T \in L(V, V)$, let $W \subset V$ be a subspace such that $T(W) \subset W$, and let $Q : V \to V/W$ be the quotient map.

(a) Show that there exists $\tilde{T} \in L(V/W, V/W)$ such that $Q \circ T = \tilde{T} \circ Q$.

(b) Assume that V is finite-dimensional and let $B = \{w_1, \ldots, w_k\}$ be a basis for W such that

$$T(w_i) = \sum_{j=1}^{i} \lambda_{ij} w_j$$

for each $i = 1, \ldots, k$ (i.e., the matrix $[T|_W]_{B,B}$ which represents $T|_W$ relative to B is upper triangular). Assume also that V/W has a basis $C = \{c_1, \ldots, c_m\}$ such that

$$\tilde{T}(c_l) = \sum_{\nu=1}^{l} \mu_{l\nu} c_\nu$$

(i.e., $[\tilde{T}]_{C,C}$ is upper triangular). Show that V has a basis A such that $[T]_{A,A}$ is upper triangular.