Mathematics 595 (CAP/TRA) Fall 2005
Homework #10 (due Friday, December 4)

Do **one** of problems #2 and #3.

A Banach space V is *strictly convex* if

$$v, v' \in V,\ v \neq v',\ ||v|| = ||v'|| = 1 \Rightarrow ||\frac{v+v'}{2}|| < 1.$$

(The geometric meaning of this condition is the following: there are no collinear triples of points on the boundary of the unit ball in V.)

V is *uniformly convex* if for all $\epsilon > 0$ there exists $\delta > 0$ so that

$$v, v' \in V,\ ||v|| = ||v'|| = 1,\ ||v - v'|| \geq \epsilon \Rightarrow ||\frac{v+v'}{2}|| < 1 - \delta.$$

V is *smooth* if for every $v_0 \in V$, $v_0 \neq 0$, there exists a unique $T_0 \in V^*$ (depending on v_0) with $||T_0|| = 1$ (operator norm) and $T_0(v_0) = ||v_0||$. (Recall that the Hahn–Banach theorem assures the existence of at least one such functional T_0.)

1. Prove that if V^* is strictly convex, then V is smooth.

2. Prove that if V^* is smooth, then V is strictly convex.

3. Prove that every finite dimensional strictly convex Banach space is uniformly convex. (Hint: in any finite-dimensional Banach space, the unit sphere $S = \{v \in V : ||v|| = 1\}$ is compact.)

4. (a) *Clarkson's inequalities* for the space ℓ^p, $1 < p < \infty$, are the following: for all $v, w \in \ell^p$,

$$||v+w||_p^p + ||v-w||_p^p \leq 2^{p-1}(||v||_p^p + ||w||_p^p), \qquad \text{if } 2 \leq p < \infty,$$

and

$$||v+w||_p^q + ||v-w||_p^q \leq 2(||v||_p^p + ||w||_p^p)^{q-1}, \qquad \text{if } 1 < p \leq 2,\ q = p/(p-1).$$

Use Clarkson's inequalities to prove that ℓ^p, $1 < p < \infty$, is uniformly convex. (You do not need to prove Clarkson's inequalities.)

(b) Give a geometric interpretation of Clarkson's inequality in the case $p = 2$.

(c)* Prove Clarkson's inequalities. (Hint: in the case $p > 2$, start by proving the real-variable inequality $|\frac{1}{2}(a+b)|^p + |\frac{1}{2}(a-b)|^p \leq \frac{1}{2}(|a|^p + |b|^p)$ for all $a, b \geq 0$.)

5*. Prove that if V is smooth and $f : V \to \mathbb{R}$ is the norm, i.e., $f(v) = ||v||$, then f has directional derivatives in every direction at every point $0 \neq v_0 \in V$, indeed,

$$D_v f(v_0) = T_0(v),$$

where T_0 is the unique linear functional corresponding to the point v_0, whose existence is asserted in the definition of smoothness.