# Mathematics 595 CAP/TRA Homework #11
## due Wednesday, 12/7

**1:** Let $(V, \langle \cdot, \cdot \rangle)$ be a Hilbert space, and let $a, b, c, d \in V$. Prove that

$$||a - c||^2 + ||b - d||^2 \leq ||a - b||^2 + ||b - c||^2 + ||c - d||^2 + ||d - a||^2.$$

What the relationship between this inequality and Clarkson's inequality in the case $V = \ell^2$?

**2:** Let $V$ be the Hilbert space $\ell^2(\mathbb{C})$ with inner product $\langle v, w \rangle = \sum_{i=1}^{\infty} v_i \overline{w_i}$. For each $v \in V$ consider the complex power series $f_v(z) = \sum_{n=0}^{\infty} v_{n+1} z^n$.

(a) Prove that the radius of convergence of $f_v$ is at least one.

(b) Fix $z \in \mathbb{C}$ with $|z| < 1$ and consider the map $T_z : V \to \mathbb{C}$ given by

$$T_z(v) = f_v(z).$$

Prove that $T_z$ is linear and bounded. Find a vector $v_z \in V$ so that $T_z(v) = \langle v, v_z \rangle$ for all $v \in V$. Compute $||T_z||^{op} = ||v_z||_2$.

**3*:** Let $T : V \to V$ be a surjective map of a Hilbert space $V$, and assume that $\langle T(v_1), T(v_2) \rangle = \langle v_1, v_2 \rangle$ for all $v_1, v_2 \in V$. Prove that $T$ is linear.