Mathematics 595 (CAP/TRA) Fall 2005
Homework #1: Solutions to selected problems

3. Let $T \in L(V,V)$, let $W \subset V$ be a subspace so that $T(W) \subset W$, and let $Q : V \to V/W$ be the quotient map.

(a) Show that there exists $\tilde{T} \in L(V/W, V/W)$ so that $Q \circ T = \tilde{T} \circ Q$.

Proof. In fact, the equation $Q \circ T = \tilde{T} \circ Q$ provides us with the **definition** of $\tilde{T}$:

$$\tilde{T}(v + W) = T(v) + W.$$

Since $Q(v) = v + W$, it's clear that $Q \circ T = \tilde{T} \circ Q$ is satisfied. But we need to check that $\tilde{T}$ is well-defined and linear.

To see that $\tilde{T}$ is well-defined, suppose that $v, v' \in V$ with $v + W = v' + W$. Then $v - v' \in W$ so $T(v - v') \in W$ and

$$\tilde{T}(v + W) = T(v) + W = T(v') + W = \tilde{T}(v' + W)$$

which shows that $\tilde{T}$ is well-defined.

Next, we show that $\tilde{T}$ is linear. Suppose that $v, v' \in V$ and $a, b \in \mathbb{F}$. Then

$$\begin{aligned} \tilde{T}(av + bv') &= T(av + bv') + W = aT(v) + bT(v') + W \\ &= a(T(v) + W) + b(T(v') + W) = a\tilde{T}(v) + b\tilde{T}(v'). \end{aligned}$$

(b) Assume that V is finite-dimensional and let $B = \{w_1, \dots, w_k\}$ be a basis for W so that

$$T(w_i) = \sum_{j=1}^{i} \lambda_{ij} w_j$$

for each $i = 1, \dots, k$ (i.e., the matrix $[T|_W]_{B,B}$ which represents $T|_W$ with respect to B is upper triangular). Assume also that V/W has a basis $C = \{c_1, \dots, c_m\}$ so that

$$\tilde{T}(c_l) = \sum_{p=1}^{l} \mu_{lp}(c_p)$$

for each $l = 1, \dots, m$ (i.e., $[\tilde{T}]_{C,C}$ is upper triangular). Show that V has a basis A so that $[T]_{A,A}$ is upper triangular.

Proof. Each element of C can be written in the form $c_l = v_l + W$ for some $v_l \in V$. (Of course, this representation is not unique.) We claim that $A = \{w_1, \dots, w_k, v_1, \dots, v_m\}$ is a basis for V so that $[V]_{A,A}$ is upper triangular.

First, let's check that A is a basis. Suppose that $a_1, \dots, a_k, b_1, \dots, b_m$ are scalars so that

$$a_1 w_1 + \dots + a_k w_k + b_1 v_1 + \dots + b_m v_m = 0.$$

Apply T to the equation to get

$$a_1T(w_1) + \cdots + a_kT(w_k) + b_1T(v_1) + \cdots + b_mT(v_m) = 0;$$

since $T(W) \subset W$ we find

$$b_1T(v_1) + \cdots + b_mT(v_m) \in W$$

and so

$$b_1\tilde{T}(c_1) + \cdots b_m\tilde{T}(c_m) = 0$$

in V/W. By the linear independence of C, all of the b_l's are zero. But then, returning to the original equation, we find

$$a_1w_1 + \cdots + a_kw_k = 0;$$

by the linear independence of B, all of the a_i's are zero. This proves that A is linearly independent.

A similar proof shows that A spans V. Let $v \in V$. Since C spans V/W we have

$$Q(v) = b_1c_1 + \cdots + b_mc_m$$

for some coefficients b_l, in other words, $v - b_1v_1 - \cdots - b_mv_m \in W$. Since B spans W,

$$v - b_1v_1 - \cdots - b_mv_m = a_1w_1 + \cdots + a_kw_k$$

which shows that A spans V.

Finally, to see the form of $[T]_{A,A}$, let's compute the action of T on the elements of A. In the first k coordinates we have

$$T(w_i) = \sum_{j=1}^{i} \lambda_{ij}w_j, \qquad i = 1, \ldots, k.$$

In the last m coordinates we have

$$\tilde{T}(c_l) = \sum_{p=1}^{l} \mu_{lp}c_p;$$

using $c_l = v_l + W$ and $Q \circ T = \tilde{T} \circ Q$ we find

$$T(v_l) - \sum_{p=1}^{l} \mu_{lp}v_p \in W.$$

Since B is a basis for W, we get

$$T(v_l) = \sum_{j=1}^{k} \lambda_{lj}w_j + \sum_{p=1}^{l} \mu_{lp}v_p$$

for some constants $\{\lambda_{lj}\}$. We see that the matrix $[T]_{A,A}$ is upper triangular, as asserted.