Mathematics 595 CAP/TRA Homework #2
due Monday, 9/12

1: A linear map $T : V \to V$ is called *nilpotent* if there exists $m \in \mathbb{N}$ so that $T^m = 0$. An $n \times n$ matrix A with entries from a field $\mathbb{F}$ is called *nilpotent* if there exists a nilpotent linear map $T : V \to V$ of an n-dimensional vector space over $\mathbb{F}$ so that $A = [T]_{B,B}$ for some basis B of V.

(a) Show that $\lambda = 0$ is the only possible eigenvalue for a nilpotent transformation T.

(b) What is the form of the general 3×3 nilpotent upper triangular matrix?

(c) Find a nilpotent 2×2 matrix $A = (a_{ij})$ over $\mathbb{R}$ with $a_{11} \neq 0$. (Hint: use a similarity.)

(d) Prove the converse to part (a): if V is finite dimensional and $\lambda = 0$ is the only eigenvalue of T, then T is nilpotent.

2: Recall that $\det : \mathbb{F}^{n \times n} = (\mathbb{F}^n)^n \to \mathbb{F}$ is the unique function satisfying the three conditions

(i) (antisymmetry) $\det(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -\det(v_1, \dots, v_j, \dots, v_i, \dots, v_n)$.

(ii) (multilinearity) $\det(av + bw, v_2, \dots, v_n) = a \det(v, v_2, \dots, v_n) + b \det(w, v_2, \dots, v_n)$.

(iii) (normalization) $\det(e_1, \dots, e_n) = 1$.

Let S_n be the group of all permutations (i.e., bijections) of the set of n elements $\{1, \dots, n\}$. For each $\pi \in S_n$, define a map $T_\pi : \mathbb{F}^n \to \mathbb{F}^n$ by setting $T_\pi(e_i) = e_{\pi(i)}$ and extending by linearity. By construction, $T_\pi \in L(\mathbb{F}^n, \mathbb{F}^n)$.

Let $B = \{e_1, \dots, e_n\}$ be the standard basis and let $A_\pi = [T_\pi]_{B,B}$.

(a) Show that the map $\epsilon : S_n \to \mathbb{F}$ given by $\epsilon(\pi) = \det(A_\pi)$ is a homomorphism of groups.

(b) Let $\pi = (ij)$ be a transposition, i.e., $\pi(i) = j$, $\pi(j) = i$, and $\pi(k) = k$ for all $k = 1, \dots, n$, $k \neq i, j$. Show that $\epsilon(\pi) = -1$.

(c) Show that every element $\pi \in S_n$ can be written as a product of transpositions $(i_1 j_1), \dots, (i_m j_m)$ (Hint: induct on n) and that $\epsilon(\pi) = (-1)^m$. Conclude that the parity of the number of transpositions in any representation of π is well-defined.

(d) Use the conditions (i)—(iii) above to deduce the representation formula for the determinant in terms of permutations: if $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\det(A) = \sum_{\pi \in S_n} \epsilon(\pi) a_{\pi(1),1} a_{\pi(2),2} \cdots a_{\pi(n),n}.$$

3: Let $V = \mathbb{F}^{\mathbb{N}} = \{x = (x_1, x_2, \dots) : x_n \in \mathbb{F}\}$ and define a transformation $T : V \to V$ by $T(x_1, x_2, x_3, \dots) = (x_2, x_3, \dots)$. T is a linear map.

(a) Which $\lambda \in \mathbb{F}$ are eigenvalues for T? For each eigenvalue, give a corresponding eigenvector.

(b) (Extra credit) Assume that $\mathbb{F} = \mathbb{R}$ and denote by

$$\ell^2 := \{x = (x_1, x_2, \dots) \in V : \sum_{n=1}^{\infty} x_n^2 < \infty\}.$$

Which $\lambda \in \mathbb{R}$ are eigenvalues for $T : \ell^2 \to \ell^2$?