Mathematics 595 (CAP/TRA) Fall 2005
Homework #3

1. Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find the Jordan canonical form of A and compute the matrix exponential e^{tA}, $t \in \mathbb{R}$.

2. (a) Show that every metric space consisting of at most three points admits an isometric embedding in $\mathbb{R}^2$.

(b) Construct a metric space of four points which does not admit an isometric embedding in $\mathbb{R}^n$ for any n.

3. In class, we considered the l^p metrics on $\mathbb{R}^n$, $1 \le p \le \infty$, and sketched the unit balls in $(\mathbb{R}^2, ||\cdot||_p)$. It is a fact that the metric spaces $(\mathbb{R}^n, ||\cdot||_p)$ and $(\mathbb{R}^n, ||\cdot||_q)$ are not isometrically equivalent unless $n = 2$ and $\{p, q\} = \{1, \infty\}$. Construct an explicit isometry mapping $(\mathbb{R}^2, ||\cdot||_1)$ onto $(\mathbb{R}^2, ||\cdot||_\infty)$. (It may help to look at the unit balls.)

4. Let (X, d) be a metric space.

(a) Let $0 < \epsilon < 1$. Show that (X, d^ϵ) is a metric space, where $d^\epsilon : X \times X \to [0, \infty)$ is the function $d^\epsilon(x, y) := d(x, y)^\epsilon$.

(b) Let X be the space of all Riemannian integrable functions $f : [0, 1] \to \mathbb{R}$ such that $\int_0^1 f(x)^2\, dx < \infty$, equipped with the metric

$$d(f, g) := \left(\int_0^1 (f(x) - g(x))^2\, dx \right)^{1/2}.$$

Show that the space $[0, 1]$ (equipped with the metric $\sqrt{|s - t|}$) admits an isometric embedding in (X, d).