Mathematics 595 (CAP/TRA) Fall 2005
Homework #3: Solutions to selected problems

2. (a) Show that every metric space consisting of at most three points admits an isometric embedding in $\mathbb{R}^2$.

High school geometry proof. Let $X = \{x,y,z\}$ with $a = d(x,y)$, $b = d(x,z)$ and $c = d(z,y)$. Assume WLOG that $a \ge b \ge c$. Construct a line segment L of length a, and mark out circles B and C centered at the vertices P and Q of this line segment with radii b and c respectively. Since $b \le a$, B intersects L at a point R. Similarly, since $c \le a$, C intersects L at a pont S.Since $b + c \ge a$, the order of these four points along segment L is P, S, R, Q. Thus $Q \in C$ is outside circle B and $S \in C$ is inside circle B; thus C must intersect B. Drawing line segments from the point of intersection to P and Q gives a triangle with side lengths a, b, c as desired.

Analytic geometry proof. Let $X = \{x,y,z\}$ and $a \ge b \ge c$ be as in the preceding proof. Let's try embedding X in $\mathbb{R}^2$ as follows: map x to $(0,0)$, map y to $(a,0)$ and map z to some point (s,t) with $0 < s < a$ and $t > 0$. We want to find s and t so that $|(s,t)-(0,0)| = b$ and $|(s,t)-(a,0)| = c$, in other words,

$$s^2 + t^2 = b^2 \quad \text{and} \quad (s-a)^2 + t^2 = c^2.$$

Subtracting these equations gives

$$a(2s - a) = b^2 - c^2$$

from which we can calculate a formula for s in terms of a, b, c.

Since $a \le b + c$ and $b \le a + c$, we have $c \ge |a - b|$ which we may rewrite in the form

$$b^2 - c^2 \le a(2b - a).$$

Combining these two equations gives $s \le b$. This shows that $t = \sqrt{b^2 - s^2}$ is well-defined.

Remark: Heron's formula gives the area of a triangle T in terms of the lengths a, b, c of the three sides. It reads

$$A(T) = \sqrt{\sigma(\sigma - a)(\sigma - b)(\sigma - c)}$$

where $\sigma = (a + b + c)/2$ is the semiperimeter. Notice that

$$A(T) = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(a+c-b)},$$

where the factors under the radical include the three *triangle inequality defects* $a + b - c$, $b + c - a$, $a + c - b$.

(b) Construct a metric space of four points which does not admit an isometric embedding in $\mathbb{R}^n$ for any n.

Proof. Let $X = \{w, x, y, z\}$ with $d(w, x) = d(x, z) = d(w, y) = d(y, z) = 1$ and $d(w, z) = d(x, y) = 2$. Suppose that X admits an isometric embedding in $\mathbb{R}^n$ for some n, and write W, X, Y, Z for the images of w, x, y, z in $\mathbb{R}^n$. Then both X and Y must be the midpoint of the line segment joining W to Z. This is clearly a contradiction, since X and Y are distinct points (in fact, $|X - Y| = 2$).

4(a). Let (X, d) be a metric space. Let $0 < \epsilon < 1$. Show that (X, d^ϵ) is a metric space, where $d^\epsilon : X \times X \to [0, \infty)$ is the function $d^\epsilon(x, y) := d(x, y)^\epsilon$.

Proof. The only difficult thing to prove is the triangle inequality. We need to prove that $d(x, y)^\epsilon \le d(x, z)^\epsilon + d(z, y)^\epsilon$ for all $x, y, z \in X$. Since d is a metric, we know that $d(x, y) \le d(x, z) + d(z, y)$. If we write $a = d(x, y)$, $b = d(x, z)$ and $c = d(z, y)$, we need to show that

$$a \le b + c \qquad \Rightarrow \qquad a^\epsilon \le b^\epsilon + c^\epsilon,$$

which will certainly follow if we can show

$$(b + c)^\epsilon \le b^\epsilon + c^\epsilon \tag{1}$$

for all $b, c \ge 0$. The case $b = 0$ or $c = 0$ is obvious, so assume $b, c > 0$. Let $t = \frac{b}{c}$ and rewrite (1) in the form

$$(1 + t)^\epsilon \le 1 + t^\epsilon \qquad \forall t > 0.$$

To prove this statement, let $f(t) = (1 + t)^\epsilon$ and $g(t) = 1 + t^\epsilon$. Then $f(0) = g(0)$ and

$$f'(t) = \epsilon(1 + t)^{\epsilon - 1} \le \epsilon t^{\epsilon - 1} = g'(t)$$

for all $t > 0$ (remember that $\epsilon < 1$), so

$$f(t) = \int_0^t f'(s)\, ds \le \int_0^t g'(s)\, ds = g(t).$$