Mathematics 595 (CAP/TRA) Fall 2005
Homework #5 (due Wednesday, October 5)

1. In class we defined the space

$$\ell^\infty = \{x = (x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}} : \sup_n |x_n| < \infty\}$$

of bounded sequences, and equipped ℓ^∞ with the maximum metric

$$d_\infty(x, y) = \sup_n |x_n - y_n|, \qquad x = (x_n), y = (y_n).$$

Show that (ℓ^∞, d_∞) is complete but not separable.

2. (a) Let (X, d) be a metric space and let $\{f_i\}_{i \in I}$ be a family (arbitrary cardinality) of real-valued L-Lipschitz functions on X. Prove that

$$F(x) := \inf_{i \in I} f_i(x)$$

is either identically equal to $-\infty$, or defines an L-Lipschitz function on X.

In (b)–(d), let (X, d) be a metric space and A be a nonempty subset of X.

(b) Let $f : A \to \mathbb{R}$ be L-Lipschitz. Prove that there is an L-Lipschitz function $F : X \to \mathbb{R}$ which extends f, i.e., $F(a) = f(a)$ for all $a \in A$. (Hint: apply (a) to the family of functions $f_a(x) = f(a) + Ld(x, a)$, $a \in A$.)

(c) Let $f : A \to \mathbb{R}^n$ be L-Lipschitz (Euclidean metric on $\mathbb{R}^n$). Prove that there is a $(\sqrt{n} \cdot L)$-Lipschitz function $F : X \to \mathbb{R}^n$ which extends f.

(d) Let $f : A \to (\ell^\infty, d_\infty)$ be L-Lipschitz. Prove that there is an L-Lipschitz function $F : X \to \ell^\infty$ which extends f.

A metric space (Y, d') is an *absolute Lipschitz retract* if whenever (X, d) is a metric space and $i : Y \to X$ is an isometric embedding, then there exists a 1-Lipschitz function $r : X \to Y$ so that $r \circ i = \text{id}$ on Y. (r is called a *retraction* of X onto Y.)

A metric space (Y, d') has the *absolute Lipschitz extension property* if whenever (X, d) is a metric space, A is a subset of X, and $f : A \to Y$ is an L-Lipschitz map, then there exists an L-Lipschitz map $F : X \to Y$ so that $F \circ i = f$ on A. (In other words, the diagram

$$\begin{array}{ccc} X & & \\ \uparrow & \searrow & \\ A & \rightarrow & Y \end{array}$$

commutes.) For example, $Y = \mathbb{R}$ and $Y = (\ell^\infty, d_\infty)$ have the absolute Lipschitz extension property (see parts (b) and (d) of the previous problem).

3*: Prove that (Y, d') is an absolute Lipschitz retract if and only if it has the absolute Lipschitz extension property. Hint: use problem 2(d) and an analog of the Frechet embedding for the space

$$\ell^\infty(Y) = \{x = (x_y)_{y \in Y} \in \mathbb{R}^Y : \sup_{y \in Y} |x_y| < \infty\}.$$