Mathematics 595 (CAP/TRA) Fall 2005
Homework #5: Solutions to selected problems

2. (a) Let (X, d) be a metric space and let $\{f_i\}_{i \in I}$ be a family of real-valued L-Lipschitz functions on X. Prove that $F(x) := \inf_{i \in I} f_i(x)$ is either identically equal to $-\infty$, or defines an L-Lipschitz function on X.

Proof: Suppose that $F(x_0) > -\infty$ for some $x_0 \in X$. Then $f_i(x_0) \ge F(x_0)$ for all i, so

$$F(x) = \inf_i f_i(x) \ge \inf_i f_i(x_0) - Ld(x, x_0) = F(x_0) - Ld(x, x_0) > -\infty$$

for all $x \in X$. For any $x, y \in X$ and any $i \in I$ we find

$$F(x) \le f_i(x) \le f_i(y) + Ld(x, y)$$

so $F(x) \le F(y) + Ld(x, y)$. Reversing the roles of x and y finishes the proof.

(b) Let A be a nonempty subset of a metric space (X, d). Let $f : A \to \mathbb{R}$ be L-Lipschitz. Prove that there is an L-Lipschitz function $F : X \to \mathbb{R}$ which extends f, i.e., $F(a) = f(a)$ for all $a \in A$.

Proof: We consider the family of functions $f_a(x) = f(a) + Ld(x, a)$, $a \in A$. We proved in class that the map $x \mapsto d(x, a)$ is 1-Lipschitz, hence f_a is L-Lipschitz. As in part (a) we define $F(x) = \inf_a f_a(x)$.

Note that $f_a(x) \ge f(a)$ for all $x \in X$ and $a \in A$, so $F(a) \ge f(a)$. On the other hand, $F(a) \le f_a(a) = f(a)$. Thus $F = f$ on A. In particular, there is a point in X at which F is not equal to $-\infty$. By part (a), F is L-Lipschitz.

(c) Let $f : A \to \mathbb{R}^n$ be L-Lipschitz (Euclidean metric on $\mathbb{R}^n$). Prove that there is a $(\sqrt{n} \cdot L)$-Lipschitz function $F : X \to \mathbb{R}^n$ which extends f.

Proof: Write $f = (f_1, \ldots, f_n)$. Each of the coordinate functions f_i is L-Lipschitz. By part (b), f_i can be extended to an L-Lipschitz function $F_i : X \to \mathbb{R}$. Then $F = (F_1, \ldots, F_n)$ extends f, and is $\sqrt{n} \cdot L$-Lipschitz, since

$$\begin{aligned} |F(x) - F(y)| &= \sqrt{|f_1(x) - f_1(y)|^2 + \cdots + |f_n(x) - f_n(y)|^2} \\ &\le \sqrt{L^2 + \cdots + L^2} = \sqrt{n} \cdot L. \end{aligned}$$

(d) Let $f : A \to (\ell^\infty, d_\infty)$ be L-Lipschitz. Prove that there is an L-Lipschitz function $F : X \to \ell^\infty$ which extends f.

Proof: Write $f = (f_1, f_2, \ldots)$ and apply part (b) to each of the coordinate functions f_i. There is no loss in the Lipschitz constant here because of the choice of the metric in ℓ^∞. (If we had used the ℓ^∞ metric in $\mathbb{R}^n$ in part (c), there would have been no loss in the Lipschitz constant.)

Remark 1. (b) is the *McShane extension theorem.* In connection with (c), *Kirzbraun's extension theorem* is worth noting: every L-Lipschitz function $f : A \to \mathbb{R}^n$ admits an L-Lipschitz extension $F : X \to \mathbb{R}^n$. (That is, the factor $\sqrt{n}$ is not needed.) The proof of this sharper result is quite a bit more complicated. Understanding which metric spaces can serve as the target in embedding theorems of this type is a question of current research interest.

A metric space (Y, d') is an *absolute Lipschitz retract* (ALR) if whenever (X, d) is a metric space and $i : Y \to X$ is an isometric embedding, then there exists a 1-Lipschitz function $r : X \to Y$ so that $r \circ i = \text{id}$ on Y.

A metric space (Y, d') has the *absolute Lipschitz extension property* (ALEP) if whenever A is a subset of (X, d) and $f : A \to Y$ is an L-Lipschitz map, then there is an L-Lipschitz map $F : X \to Y$ so that $F \circ i = f$ on A.

3*. Prove that (Y, d') is an ALR if and only if it has the ALEP.

Proof: Suppose that (Y, d') has the ALEP. Let (X, d) contain an isometric copy of (Y, d'). Apply the ALEP condition to fill in the diagram

$$\begin{array}{ccc} X & & \\ \uparrow & \searrow & \\ Y & \stackrel{\text{id}}{\to} & Y \end{array}$$

The resulting map from X to Y is the desired retraction.

Now suppose that (Y, d') is an ALR. Let $A \subset X$ and let $f : A \to Y$ be an L-Lipschitz map. The Frechet embedding maps Y isometrically into $\ell^\infty(Y)$. The resulting L-Lipschitz map from A to $\ell^\infty(Y)$ can be extended to an L-Lipschitz map F from X to $\ell^\infty(Y)$ by part (d) of the previous problem. Since Y is an ALR, there is a 1-Lipschitz retraction $r : \ell^\infty(Y) \to Y$. Then $r \circ F : X \to Y$ is an L-Lipschitz map extending $f : A \to Y$. The entire proof is summarized in the following diagram:

$$\begin{array}{ccc} X & \stackrel{F}{\to} & \ell^\infty(Y) \\ \uparrow & & \uparrow\downarrow \\ Y & \stackrel{f}{\to} & Y \end{array}$$