Mathematics 595 (CAP/TRA) Fall 2005
Homework #6 (due Wednesday, October 19)

1. (a) Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be one-to-one and onto. Prove that the following are equivalent:

(i) f^{-1} is continuous,

(ii) f is *open*, i.e., U open in X implies $f(U)$ open in Y,

(iii) f is *closed*, i.e., F closed in X implies $f(F)$ closed in Y.

(b) Let $(X, \mathcal{T})$ be a compact topological space, let $(Y, \mathcal{T}')$ be a metrizable topological space ($\mathcal{T}' = \mathcal{T}_{d'}$ is the topology induced by a metric d' on Y) and let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be continuous, one-to-one and onto. Prove that f^{-1} is continuous.

Extra credit: What is the most general condition on $(Y, \mathcal{T}')$ under which this conclusion holds?

2. A topological space $(X, \mathcal{T})$ is *discrete* if $\mathcal{T}$ is the set of all subsets of X, i.e., every set in X is open.

Prove that $(X, \mathcal{T})$ is discrete if and only if every map from $(X, \mathcal{T})$ to another topological space $(Y, \mathcal{T}')$ is necessarily continuous.

3. Let $(X, \mathcal{T})$ be a topological space and let $\sim$ be an equivalence relation on X. Let $Y = X/\sim$ be the quotient space, and let $\mathcal{T}'$ be the collection of all subsets U of Y with the following property: U is a collection of equivalence classes $[x]$ such that $\bigcup_{[x]\in U}\{x\}$ is an open set in X.

(a) Prove that $\mathcal{T}'$ is a topology on Y. It is called the *quotient topology.*

(b) Prove that $Q : (X, \mathcal{T}) \to (Y, \mathcal{T}')$, $Q(x) = [x]$ is continuous.

(c) Let $(Z, \mathcal{T}'')$ be another topological space, and let $f : X \to Z$ be a continuous map which is constant on equivalence classes ($x \sim y \Rightarrow f(x) = f(y)$). Prove that there exists a unique continuous map $g : Y \to Z$ so that $g \circ Q = f$.

(d) Let $X = \mathbb{R}^2$ with the standard topology, and define the equivalence relation $(x, y) \sim (x', y')$ if and only if $(x' - x, y' - y) \in \mathbb{Z} \times \mathbb{Z}$. Identify the quotient space $\mathbb{R}^2/\sim$ up to homeomorphism. (No proof needed.)

(e) Repeat (d) for $X = \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ with the standard topology, and the equivalence relation $z \sim z'$ if and only if $z = z'$ or $z = -z'$.

(f)* Repeat (d) for $X = \mathbb{S}^2 = \{p \in \mathbb{R}^3 : |p| = 1\}$ with the standard topology, and the equivalence relation $p \sim p'$ if and only if $p = p'$ or $p = -p'$.

4. Let A and B be disjoint subsets of a metric space (X, d), with A closed and B compact. Prove that $\operatorname{dist}(A, B) > 0$. (See Exam 1 for the definition of the distance between sets.)

5.* Let (X, d) be a compact metric space.

(a) Let $f : X \to X$ be a 1-*co-Lipschitz* map: $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. Prove that f is an isometry.

(b) Let $f : X \to X$ be 1-Lipschitz and onto. Prove that f is an isometry.