

Mathematics 595 (CAP/TRA) Fall 2005

Homework #7 (due Friday, October 28)

1. (a) Prove the *discrete Chebyshev inequality*:

$$\#\{n \in \mathbb{N} : |v_n| > \lambda\} \leq \left(\frac{\|v\|_p}{\lambda}\right)^p$$

for all $v = (v_n) \in \ell^p$ and all $\lambda > 0$. Here $\#S$ denotes the cardinality of a set S .

- (b) Define the space $\text{weak-}\ell^p$ of *weakly ℓ^p functions* to be

$$\text{weak-}\ell^p := \{v = (v_n) \in \mathbb{R}^{\mathbb{N}} : \sup_{\lambda > 0} \lambda^p \#\{n : |v_n| > \lambda\} < \infty\}.$$

By part (a), $\ell^p \subset \text{weak-}\ell^p$. Prove that this inclusion is strict.

2. Let V be a normed vector space and let W be a subspace of V . Let

$$W^\perp = \{T \in V^* : T|_W = 0\}$$

be the *annihilator* of W .

- (a) Prove that V^*/W^\perp is isometrically isomorphic with W^* . (Hint: use the Hahn–Banach theorem.)

- (b) Prove that $(V/W)^*$ is isometrically isomorphic with W^\perp . (Hint: use the canonical quotient map $Q : V \rightarrow V/W$.)

3. Let $V = C^k([0, 1])$, $k \in \mathbb{N}$. Prove that the two norms

$$\|f\|_1 := \sum_{i=0}^k \max_{0 \leq x \leq 1} |f^{(i)}(x)|$$

and

$$\|f\|_2 := \sum_{i=0}^{k-1} |f^{(i)}(0)| + \max_{0 \leq x \leq 1} |f^{(k)}(x)|$$

are equivalent, i.e., there exists a constant C so that

$$\frac{1}{C} \|f\|_1 \leq \|f\|_2 \leq C \|f\|_1$$

for all $f \in V$.

4. Let $(V_i, \|\cdot\|_i)$ be a sequence of normed vector spaces, let $V = \prod_i V_i$ be the product vector space, and let $\|\cdot\|_p$, $1 \leq p \leq \infty$ be the ℓ^p norm on the subspace $\oplus_p V_i \subset V$ (as defined in Exercise 3.2.15). For each i , let $\|\cdot\|_i^{op}$ be the operator norm on V_i^* , and let $\|\cdot\|_p^{op}$ be the corresponding product norm on $\oplus_q V_i^* \subset V^* = \prod_i V_i^*$.

For $1 \leq p < \infty$, prove that the space $(\oplus_p V_i, \|\cdot\|_p)^*$ is isometrically isomorphic with $(\oplus_q V_i^*, \|\cdot\|_q^{op})$, $\frac{1}{p} + \frac{1}{q} = 1$.