# Mathematics 595 (CAP/TRA) Fall 2005
## Homework #7: Solutions to selected problems

**2.** Let $V$ be a normed vector space and let $W$ be a subspace of $V$. The *annihilator* of $W$ is the subspace $W^\perp = \{T \in V^* : T|_W = 0\}$. (a) Prove that $V^*/W^\perp$ is isometrically isomorphic with $W^*$. (b) Prove that $(V/W)^*$ is isometrically isomorphic with $W^\perp$.

*Proof of (a).* Define $\Phi(T + W^\perp) = T|_W$. Here $T|_W$ denotes the restriction of the map $T : V \to \mathbb{R}$ to the subspace $W$. It's easy to see that $\Phi$ is well-defined and linear. Since $T|_W = 0$ if and only if $T \in W^\perp$, $T$ has trivial kernel and hence is injective. We estimate

$$|\Phi(T + W^\perp)(w)| = |T(w)| \le ||T + U||\,||w||$$

for any $U \in W^\perp$. Taking the infimum over all $U$, we see that $\Phi(T + W^\perp) \in W^*$ with $||\Phi(T + W^\perp)|| \le ||T + W^\perp||$. Thus $\Phi : V^*/W^\perp \to W^*$ with $||\Phi|| \le 1$.

On the other hand, if $S \in W^*$ then we may choose $T = T_S \in V^*$ so that $T|_W = S$ and $||T|| = ||S||$ (Hahn–Banach theorem). Define $\Psi(S) = Q \circ T_S$, where $Q : V^* \to V^*/W^\perp$ is the quotient map. Note that

$$||\Psi(S)|| = ||Q \circ T_S|| \le ||Q||\,||T_S|| \le ||S||,$$

so $\Psi(S) \in V^*/W^\perp$, i.e., $\Psi : W^* \to V^*/W^\perp$. Since

$$\Phi \circ \Psi(S) = \Phi(Q \circ T_S) = \Phi(T_S + W^\perp) = (T_S)|_W = S,$$

we conclude that $\Phi$ is onto, and $||\Phi|| = 1$ (Why?). To summarize, $\Phi$ defines an isometric isomorphism from $V^*/W^\perp$ to $W^*$ which is onto.

*Proof of (b).* For $T \in (V/W)^*$ define $\Phi(T) = T \circ Q$, where $Q : V \to V/W$ is the canonical quotient map. Then $\Phi(T)|_W = T(0) = 0$, so $\Phi : (V/W)^* \to W^\perp$. It's easy to see that $\Phi$ is linear. Since

$$||\Phi(T)|| = ||T \circ Q|| \le ||T||\,||Q|| \le ||T||$$

we see that $\Phi$ is bounded with $||\Phi|| \le 1$.

On the other hand, if $S \in W^\perp$ define $T = T_S : V/W \to \mathbb{R}$ by $T(v + W) = S(v)$ and set $\Psi(S) = T_S$. The map $T$ is well-defined since $S|_W = 0$. It is also clearly linear. Note that

$$||\Psi(S)|| = ||T|| = \sup_{v+W:||v+W||\le 1} ||T(v + W)|| \le \sup_{v:||v||\le 1} ||S(v)|| = ||S||.$$

(To see why the middle inequality holds, observe that we may choose a sequence of representatives $v + w_n \in v + W$ with $||v + w_n|| \le 1 + \frac{1}{n}$.) Since

$$\Phi \circ \Psi(S) = \Phi(T) = T \circ Q = S$$

we conclude that $\Phi$ is onto, and $||\Phi|| = 1$. Again in summary, $\Phi$ defines an isometric isomorphism from $(V/W)^*$ to $W^\perp$.

**4.** Let $(V_i, ||\cdot||_i)$ be a sequence of normed vector spaces, let $V = \prod_i V_i$ be the product vector space, and let $||\cdot||_p$, $1 \leq p \leq \infty$ be the $\ell^p$ norm on the subspace $\oplus_p V_i \subset V$ (as defined in Exercise 3.2.15). For each $i$, let $||\cdot||_i^{op}$ be the operator norm on $V_i^*$, and let $||\cdot||_p^{op}$ be the corresponding product norm on $\oplus_q V_i^* \subset V^* = \prod_i V_i^*$. For $1 \leq p < \infty$, prove that $(\oplus_p V_i, ||\cdot||_p)^*$ is isometrically isomorphic with $(\oplus_q V_i^*, ||\cdot||_q^{op})$, $\frac{1}{p} + \frac{1}{q} = 1$.

*Proof.* Given an element $(T_i)$ in $\oplus_q V_i^*$, define $S = S_{(T_i)} : \oplus_p V_i \to \mathbb{R}$ by

$$S((v_i)) = \sum_i T_i(v_i)$$

(by convention, the sum is from 1 to infinity if the limits are not specified). This is clearly linear, moreover it is bounded since

$$\begin{aligned} |S((v_i))| &\leq \sum_i |T_i(v_i)| \leq \sum_i ||T_i||\,||v_i|| \\ &\leq \left(\sum_i ||T_i||^q\right)^{1/q} \left(\sum_i ||v_i||^p\right)^{1/p} = ||(T_i)||_q ||(v_i)||_p \end{aligned}$$

by the countably infinite version of Hölder's inequality. Thus $S \in (\oplus_p V_i)^*$ with $||S|| \leq ||(T_i)||_q$.

The other direction is rather more complicated. Given $T \in (\oplus_p V_i)^*$ we must identify the appropriate element of $\oplus_q V_i^*$ and show that the norms behave appropriately by testing on a suitable element of $\oplus_p V_i$; this involves a further choice of "good" points $u_i \in V_i$. Here are the details.

For each index $i$, we define a map $T_i : V_i \to \mathbb{R}$ by $T_i(v_i) = T(0, \ldots, 0, v_i, 0, \ldots)$, where $v_i$ occurs in the $i$th position. This is linear, and bounded since $||T_i|| \leq ||T||$. Thus $T_i \in V_i^*$, and we may consider the sequence $(T_i)$. We claim that it is in $\oplus_q V_i^*$ with $||(T_i)||_q \leq ||T||$.

We want to test the boundedness of $T$ by applying it to a suitable element of $\oplus_p V_i$. We consider a sequence $(c_j)$ of nonnegative real numbers and a sequence $(u_j)$ where $u_j \in V_j$. (The exact choice of $c_j$ and $u_j$ will be given later.) Fixing $N \in \mathbb{N}$, we apply $T$ to $u = (\pm c_1 u_1, \ldots, \pm c_N u_N, 0, \ldots)$, where the sign of the $j$th term is chosen to be the sign of $T_j(u_j)$. Since $T$ is bounded, we find

$$\left|\sum_{i=1}^N \pm c_i T_i(u_i)\right| = \left|\sum_{i=1}^N T_i(\pm c_i u_i)\right| \leq ||T||||u||_p$$

which reads

$$\sum_{i=1}^N c_i |T_i(u_i)| \leq ||T|| \left(\sum_{i=1}^N c_i^p ||u_i||^p\right)^{1/p}.$$

At this point we can see how to choose the sequence $(c_j)$; let

$$c_i = \frac{|T_i(u_i)|^{q-1}}{||u_i||^q}.$$

This choice makes the summands on each side identical:

$$c_i|T_i(u_i)| = c_i^p||u_i||^p = \frac{|T_i(u_i)|^q}{||u_i||^q}.$$

We find

$$\sum_{i=1}^N \frac{|T_i(u_i)|^q}{||u_i||^q} \leq ||T|| \left( \sum_{i=1}^N \frac{|T_i(u_i)|^q}{||u_i||^q} \right)^{1/p}$$

or

$$\left( \sum_{i=1}^N \frac{|T_i(u_i)|^q}{||u_i||^q} \right)^{1/q} \leq ||T||.$$

We still have not stated explicitly which sequence $(u_i)$ we are using. Fix $\epsilon > 0$ and choose $0 \neq u_i \in V_i$ so that

$$\frac{|T_i(u_i)|}{||u_i||} > (1-\epsilon)||T_i||.$$

We find

$$(1-\epsilon) \left( \sum_{i=1}^N ||T_i||^q \right)^{1/q} \leq ||T||.$$

Letting $N \to \infty$ and $\epsilon \to 0$ gives

$$||(T_i)||_q = \left( \sum_i ||T_i||^q \right)^{1/q} \leq ||T||.$$

The operations $T \mapsto (T_i)$ and $(T_i) \mapsto S_{(T_i)}$ (from the first part) are inverses. We conclude that $(\oplus_p V_i)^*$ and $\oplus_q V_i^*$ are isometrically isomorphic.