Mathematics 595 (CAP/TRA) Fall 2005
Homework #8 (due Friday, November 4)

1. Let V be a normed vector space, let $v_1, \ldots, v_n$ be a linearly independent set of vectors in V, and let $c_1, \ldots, c_n$ be scalars. Prove that there exists $T \in V^*$ so that $T(v_i) = c_i$ for all $i = 1, \ldots, n$.

2. If V is a normed vector space and $v \in V$, prove that

$$||v|| = \sup\{|T(v)| : T \in V^*, ||T||^{op} \le 1\},$$

where the norm on V^* is the operator norm.

3. (a) If $v = (v_n) \in \ell^\infty$ and L is a Banach limit, prove that

$$\inf_n v_n \le L(v) \le \sup_n v_n.$$

(b) If v and L are as in part (a), prove that

$$\liminf_n v_n \le L(v) \le \limsup_n v_n.$$

(c) Prove that no Banach limit is multiplicative, in fact, that there exist two sequences $v, w \in \ell^\infty$ so that $L(vw) \neq L(v)L(w)$ for any Banach limit L, where $vw = (v_1 w_1, v_2 w_2, \ldots)$.

4*. (a) Prove that c_0 and c are separable.

(b) Let ac denote the space of almost convergent sequences (defined in the notes). Notice that ac contains the separable space c, and is contained in the nonseparable space ℓ^∞. Is ac separable or not?

5*. Prove that $v \in \ell^\infty$ is almost convergent to a limit L if and only if

$$\lim_{p \to \infty} \frac{v_n + \cdots + v_{n+p-1}}{p} = L$$

uniformly in n.

Hint: consider the operators $p^\pm : \ell^\infty \to \mathbb{R}$ given by

$$p^-(v) = \sup_F \liminf_{k \to \infty} \sum_{i \in F} v_{i+k}$$

and

$$p^+(v) = \inf_F \limsup_{k \to \infty} \sum_{i \in F} v_{i+k},$$

where the infimum and supremum are taken over all finite subsets of $\mathbb{N}$. Use the Hahn–Banach theorem.