Mathematics 595 (CAP/TRA) Fall 2005
Homework #9 (due Friday, November 11)

A set S in a vector space V is *convex* if for all $v, v' \in S$ the line segment

$$[v, v'] = \{tv' + (1-t)v : 0 \le t \le 1\}$$

lies in S. A set S in a normed vector space V is *path connected* if for all $v, v' \in \Omega$ there is a continuous function $\gamma : [0,1] \to \Omega$ with $\gamma(0) = v$ and $\gamma(1) = v'$.

1. (a) Let V, W be Banach spaces and let $\Omega \subset V$ be open and convex. Let $f : \Omega \to W$ be differentiable at all points of Ω, with $Df(v_0) = 0$ for all $v_0 \in \Omega$. Prove that f is constant.

(b) Repeat part (a) with "convex" replaced by "path connected".

2. Let V and W be Banach spaces, let Ω be an open set in V, and let $[v, v'] \subset \Omega$. If $f : \Omega \to W$ is differentiable at all points of Ω, prove that

$$||f(v') - f(v) - Df(v_0)(v' - v)|| \le ||v - v'|| \sup_{0 \le t \le 1} ||Df(tv' + (1-t)v) - Df(v_0)||$$

for all $v_0 \in \Omega$.

3. Let $T : C[0,1] \to C^1[0,1]$ be the linear map $T(f)(x) = \int_0^x f(t)\, dt$.

(a) Prove that $T \in L(C[0,1], C^1[0,1])$ with $||T|| = 1$.

(b) Show that there exists a linear map $S : C^1[0,1] \to C[0,1]$ which is a right inverse for T: $S \circ T = \mathrm{Id}$.

(c) Show that T has no *bounded* linear right inverse, i.e., the map S from part (b) cannot be chosen in $L(C^1[0,1], C[0,1])$.

The *directional derivative* of $f : V \to W$ at $v_0 \in V$ in the direction v is

$$D_v f(v_0) = \lim_{t \to 0} \frac{1}{t}(f(v_0 + tv) - f(v_0)),$$

if the limit exists.

4. Prove that if $f : V \to W$ is differentiable at v_0, then the directional derivative at v_0 exists in every direction v, and $D_v f(v_0) = Df(v_0)(v)$.