Mathematics 595 CAP/TRA Project #1
due Friday, 9/30

Do **one** of the following projects.

Project #1A: Non-commutative matrix exponentiation

Recall that the matrix exponential e^A for an $n \times n$ matrix A is defined as

$$e^A = \sum_{j=0}^{\infty} \frac{1}{j!} A^j.$$

When A and B commute, we have

$$e^{A+B} = e^A e^B.$$

We denote by $M(n, \mathbb{R})$ the space of $n \times n$ matrices with real entries.

1. Find matrices A and B so that $e^{A+B} \neq e^A e^B$.

2. Identify the function $\delta_1 : M(3, \mathbb{R}) \times M(3, \mathbb{R}) \to M(3, \mathbb{R})$ which makes the following equation true for all upper triangular nilpotent matrices $A, B \in M(3, \mathbb{R})$:

$$e^A e^B = e^{A+B+\delta_1(A,B)}.$$

3. Identify the function $\delta_2 : M(4, \mathbb{R}) \times M(4, \mathbb{R}) \to M(4, \mathbb{R})$ which makes the following equation true for all upper triangular nilpotent matrices $A, B \in M(4, \mathbb{R})$:

$$e^A e^B = e^{A+B+\delta_2(A,B)}.$$

   What is the relationship between the functions δ_1 and δ_2?

4. In the literature you can find descriptions of the Baker–Campbell–Hausdorff (BCH) formula. State and discuss this formula. What is the relationship between the BCH formula and your answers in parts 2 and 3?

   Give a general description of what the analog of your formulas from parts 2 and 3 should look like for (a) arbitrary upper triangular nilpotent matrices, (b) arbitrary matrices.

Project #1B: Free vector spaces and tensor products

Let I be a set (of arbitrary cardinality) and let $\mathbb{F}$ be a field. The *free vector space* over I is the vector subspace of $\mathbb{F}^I$ given by

$$\mathrm{Free}(I,\mathbb{F}) = \{v = (v_i)_{i\in I} \in \mathbb{F}^I : (v_i) \text{ is essentially zero}\}.$$

(Recall that a collection $(v_i)_{i\in I}$ is called essentially zero if $v_i \neq 0$ for only finitely many $i \in I$.)

$\mathrm{Free}(I,\mathbb{F})$ is a vector space over $\mathbb{F}$ with basis $(e^i)_{i\in I}$, where $e^i \in \mathrm{Free}(I,\mathbb{F})$ is given by $(e^i)_j = 1$ if $i = j$ and $(e^i)_j = 0$ if $i \neq j$ (Kronecker delta function).

For example, if I is a finite set, say, of cardinality n, then $\mathrm{Free}(I,\mathbb{F}) = \mathbb{F}^n$, and the basis is the standard basis.

1. Let $T \in L(\mathrm{Free}(\mathbb{N},\mathbb{F}), \mathrm{Free}(\mathbb{N},\mathbb{F}))$ be the shift transformation $T(v)_i = v_{i+1}$. Which values $\lambda \in \mathbb{F}$ are eigenvalues for T? For each such eigenvalue, give a (nonzero) eigenvector.
2. Let I be an arbitrary index set. For each function $f : I \to I$, define a linear transformation $T_f : \mathbb{F}^I \to \mathbb{F}^I$ by setting $T(v)_i = v_{f(i)}$ and extending linearly. Characterize those functions $f : I \to I$ for which $T_f : \mathrm{Free}(I,\mathbb{F}) \to \mathrm{Free}(I,\mathbb{F})$.
3. Let V be a vector space over $\mathbb{F}$, and let $g : I \to V$ be any function. Show that there exists a unique function $G \in L(\mathrm{Free}(I,\mathbb{F}), V)$ so that
$$g(i) = G(e^i) \quad \forall\, i \in I.$$
4. Let V and W be vector spaces. We can identify the free vector space $\mathrm{Free}(V \times W,\mathbb{F})$ over $V \times W$ with the set of all formal (finite) linear combinations of basis elements of the form $e^{(v,w)}$, $(v,w) \in V \times W$. Explain this statement.

Using the identification in part 4, consider the set of linear relations

$$e^{(av_1+bv_2,w)} - ae^{(v_1,w)} - be^{(v_2,w)} \quad \text{and} \quad e^{(v,aw_1+bw_2)} - ae^{(v,w_1)} - be^{(v,w_2)}, \tag{1}$$

where $a, b \in \mathbb{F}$, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$. (Note that these are **nonzero** elements of $\mathrm{Free}(I,\mathbb{F})$!) The *tensor product* of V and W is the space obtained by factoring over these relations. More precisely, it is the quotient space

$$V \otimes W = \mathrm{Free}(V \times W,\mathbb{F})/Z.$$

where Z denotes the span of the set of all elements of $\mathrm{Free}(I,\mathbb{F})$ of the form in (1). The usual notation for the elements of this space is $v \otimes w = e^{(v,w)} + Z$.

5. If B is a basis for V and C is a basis for W, show that $\{b \otimes c : b \in B, c \in C\}$ is a basis for $V \otimes W$. (This is a good exercise in working with quotient spaces and the tensor product.)
6. What else can you find out from the literature about free vector spaces and tensor products?