Math 595 (CAP/TRA) Fall 2005
Project 2: Hausdorff metric on compacta hyperspace
due Monday, November 14

Let (X, d) be a metric space and let $CL(X)$ denote the collection of all nonempty closed subsets of X. For $A \in CL(X)$ and $\epsilon > 0$, define the *ϵ-neighborhood of A*:

$$N_\epsilon(A) = \{x \in X : \mathrm{dist}(x, A) < \epsilon\} = \bigcup_{x \in A} B(x, \epsilon).$$

(See Exam 1 for the definition of $\mathrm{dist}(x, A)$.)

(a) Let $A, B \in CL(X)$. Prove that

$$\inf\{\epsilon > 0 : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\} = \max\{\sup_{a \in A} \mathrm{dist}(a, B), \sup_{b \in B} \mathrm{dist}(b, A)\}.$$

Denote this common value by $D(A, B)$.

(b) What is the value of $D(A, B)$ when $X = \mathbb{R}^2$ and A and B are two parallel line segments of equal length? Does your answer depend on the length of the segments? What is the value of $D(A, B)$ when A and B are two parallel (infinite) lines? Give an example to show that $D(A, B)$ can be infinite.

(c) Show that $D(A, B) < \infty$ if A and B are compact. Let $\mathcal{K}(X) \subset CL(X)$ be the collection of all nonempty compact sets in X. Prove: D is a metric on $\mathcal{K}(X)$.

$\mathcal{K}(X)$ is the *(compacta) hyperspace* of (X, d), and D is the *Hausdorff metric*.

(d) Show that $x \mapsto \{x\}$ defines an isometric embedding of (X, d) into $(\mathcal{K}(X), D)$.

(e) Prove that $\mathrm{diam} : (\mathcal{K}(X), D) \to \mathbb{R}$ is 2-Lipschitz.

(f) (**Extra credit**) Prove that $(\mathcal{K}(X), D)$ is complete if (X, d) is complete.

Hint: Let (A_n) be Cauchy in $(\mathcal{K}(X), D)$. Let A be the set of $x \in X$ for which every ball $B(x, \epsilon)$, $\epsilon > 0$, meets infinitely many sets A_n. Prove that A_n converges to A.

The next three parts deal with the following situation: (X, d) is a complete metric space, and $f_1, \ldots, f_M : X \to X$ are contractive self-maps of X, i.e., f_i is k_i-Lipschitz for some $k_i < 1$.

(g) Prove that $F : \mathcal{K}(X) \to \mathcal{K}(X)$ given by

$$F(A) = f_1(A) \cup \cdots \cup f_M(A)$$

is k-Lipschitz with respect to D, where $k = \max\{k_1, \ldots, k_M\}$. Conclude from part (f) and the Contraction Mapping Principle that there exists a unique set K in $\mathcal{K}(X)$ satisfying $F(K) = K$.

The collection $\mathcal{F} = \{f_1, \ldots, f_M\}$ is called an *iterated function system*, and the set K whose existence is guaranteed in the previous paragraph is called the *invariant set* associated with $\mathcal{F}$.

(h) Let $W = \{1, \ldots, M\}$ and let $\Sigma = W^{\mathbb{N}} = \{w = (w_1, w_2, \ldots) : w_m \in W\}$. For each $w \in W$, prove that $\lim_{m\to\infty} f_{w_1} \circ \cdots \circ f_{w_m}(x_0)$ exists as a point in X, and is independent of the choice of x_0.

(i) Define a map $\pi : \Sigma \to X$ by $\pi(w) = \lim_{m\to\infty} f_{w_1} \circ \cdots \circ f_{w_m}(x_0)$ for any $x_0 \in X$. Equip W with the discrete topology, and equip Σ with the Tychonoff (product) topology. Prove that π is continuous.

(Hint: the topology on Σ is metrizable, e.g., it is generated by the metric

$$d(v, w) = 2^{-M(v,w)}, \qquad v, w \in \Sigma, M(v, w) := \min\{m : v_m \neq w_m\}.$$

Thus it suffices to prove that π is sequentially continuous.)

Many classical self-similar fractal sets in $\mathbb{R}^n$ can be described as invariant sets for iterated function systems.

(j) Identify the invariant sets associated with the following iterated function systems:

(i) $(X, d) = \mathbb{R}$, $\mathcal{F} = \{f_1, f_2\}$, $f_1(x) = \frac{1}{2}x$, $f_2(x) = \frac{1}{2}x + \frac{1}{2}$.
(ii) $(X, d) = \mathbb{R}$, $\mathcal{F} = \{f_1, f_2\}$, $f_1(x) = \frac{1}{3}x$, $f_2(x) = \frac{1}{3}x + \frac{2}{3}$.
(iii) $(X, d) = \mathbb{R}^2 = \mathbb{C}$, $\mathcal{F} = \{f_1, f_2, f_3\}$,

$$f_1(z) = \frac{1}{2}z + \frac{1}{2}, \quad f_2(z) = \frac{1}{2}z + \frac{1}{2}e^{2\pi i/3}, \quad f_3(z) = \frac{1}{2}z + \frac{1}{2}e^{-2\pi i/3}$$

(using complex notation). Draw a sketch of the invariant set in case (iii).

(k) What else can you find out about hyperspaces and/or the Hausdorff metric?

(Suggestions: more about iterated function systems, Gromov–Hausdorff distance between metric spaces, ...)

The function $D : CL(X) \times CL(X) \to [0, \infty]$ is an infinite-valued metric (see section 2.1). The space $CL(X)$ separates into various components, each consisting of elements with mutually finite Hausdorff distance. Let $\mathcal{K}(X; C_0)$ be the collection of all elements C of $CL(X)$ which are at a finite Hausdorff distance from a fixed element $C_0 \in CL(X)$. For example, when $X = \mathbb{R}^n$, $\mathcal{K}(\mathbb{R}^n) = \mathcal{K}(\mathbb{R}^n; \{x_0\})$ for some (any) $x_0 \in \mathbb{R}^n$ (this is essentially the content of the Heine–Borel theorem).

(l) (**Extra credit**) Prove that ℓ^∞ embeds isometrically in the hyperspace $\mathcal{K}(\mathbb{R}, C_0)$, where $C_0 = \{3n^2 : n = 1, 2, \ldots\}$. Conclude (by the Kuratowski embedding theorem) that every separable metric space can be isometrically embedded in $\mathcal{K}(\mathbb{R}, C_0)$, i.e.,

> Every separable metric space can be realized isometrically as a collection of closed subsets of the real line, equipped with the Hausdorff metric.

Hint: For $n \in \mathbb{N}$, define $T_n : \mathbb{R} \to \mathbb{R}$ by $T_n(x) = \min\{n, \max\{x, -n\}\}$. For $x = (x_n) \in \ell^\infty$, set $x_{ij} = T_j(x_i)$. Prove that the map $\Phi : \ell^\infty \to \ell^\infty$ given by

$$\Phi(x) = (x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, \ldots)$$

is an isometric embedding of ℓ^∞ in itself. It suffices to embed $\Phi(\ell^\infty)$ in $\mathcal{K}(\mathbb{R}, C_0)$.