Math 230 Spring 2005
Section $7.2\frac{1}{2}$: Areas of Surfaces of Revolution

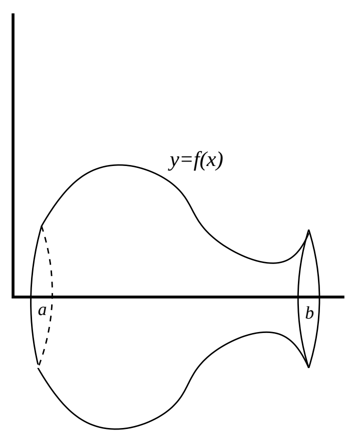
In section 7.2 you learned how to compute the volume of a solid given by revolving a region defined by the graphs of functions about an axis. Now we'll discuss how to compute the area of the surface which bounds such a solid. In particular, we'll show how to rederive some basic formulas from geometry for the surface areas of cylinders, cones, etc.

Recall that the volume of the solid S obtained by revolving the graph of $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is

$$\text{Volume}(S) = \pi \int_a^b f(x)^2 \, dx.$$

Provided that $f(x)$ is nonnegative for all $a \leq x \leq b$, the surface area of this solid is

$$\text{SurfArea}(S) = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2} \, dx. \tag{1}$$



Later on in these notes, we'll find out why equation (1) is true. First, let's look at a few examples.

Example 1. Let $f(x) = r$ be a constant function. Then the solid S is a *cylinder* whose base is a circle of radius r and whose height is $h = b - a$. The volume of S is

$$\text{Volume}(S) = \pi \int_a^b r^2 \, dx = \pi r^2 (b - a) = \pi r^2 h,$$

while the surface area is

$$\text{SurfArea}(S) = 2\pi \int_a^b r\sqrt{1 + 0} \, dx = 2\pi r(b - a) = 2\pi rh.$$

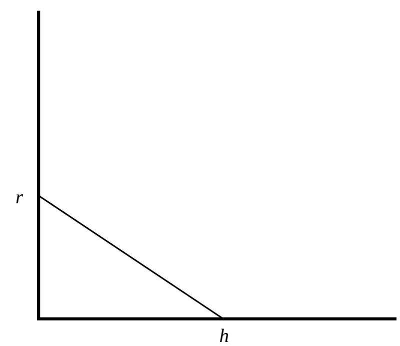
Example 2. Let $a = 0$ and $b = h$ and let $f(x) = r(1 - \frac{x}{h})$. Then the solid S is a *cone* whose base is a circle of radius r and whose height is h. The volume of S is

$$\text{Volume}(S) = \pi \int_0^h r^2 (1 - \frac{x}{h})^2 \, dx = \frac{1}{3} \pi r^2 h$$

(we discussed this example in section 7.2). Since $f'(x) = -\frac{r}{h}$ for all x, the surface area is

$$\begin{aligned} \text{SurfArea}(S) &= 2\pi \int_0^h r(1 - \frac{x}{h}) \sqrt{1 + (-\frac{r}{h})^2} \, dx = 2\pi r \sqrt{1 + \frac{r^2}{h^2}} \int_0^h (1 - \frac{x}{h}) \, dx \\ &= 2\pi r \sqrt{1 + \frac{r^2}{h^2}} \, (x - \frac{x^2}{2h}) \Big|_0^h = 2\pi r \sqrt{1 + \frac{r^2}{h^2}} \cdot \frac{h}{2} = \pi r \sqrt{h^2 + r^2}. \end{aligned}$$

Notice that $\sqrt{h^2 + r^2}$ is the *slant height* of the cone, i.e., the length of the hypotenuse of the right triangle with vertices $(0, 0)$, $(0, r)$ and $(h, 0)$.

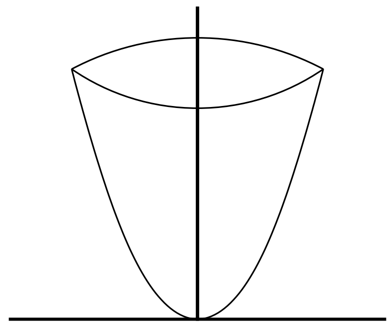


These formulas for the volume and surface area of a circular cylinder and circular cone are standard facts from elementary geometry; what is new here is the *derivation* of these formulas via integration.

The formula for the area of a surface of revolution about the y-axis is similar. If $x = g(y)$ is defined and nonnegative for $c \le y \le d$, then the area of the surface S obtained by revolving the graph of g about the y-axis is

$$\text{SurfArea}(S) = 2\pi \int_c^d g(y) \sqrt{1 + g'(y)^2} \, dy.$$

Example 3. Find the surface area of the paraboloid S obtained by revolving the graph of $y = x^2$ between $x = 0$ and $x = \sqrt{3}$ about the y-axis.



Solution. We write $x = g(y) = \sqrt{y}$ and observe that the range of y-values is from $y = 0$ to $y = 3$. Since $g'(y) = 1/(2\sqrt{y})$ we find

$$\begin{aligned}
\text{SurfArea}(S) &= 2\pi \int_0^3 \sqrt{y}\sqrt{1 + (\frac{1}{2\sqrt{y}})^2}\, dy = 2\pi \int_0^3 \sqrt{y}\sqrt{1 + \frac{1}{4y}}\, dy \\
&= 2\pi \int_0^3 \sqrt{y + \frac{1}{4}}\, dy = \frac{4\pi}{3}\,(y + \frac{1}{4})^{3/2}\Big|_0^3 = \frac{4\pi}{3}\left[\left(\frac{13}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right] \\
&= \frac{4\pi}{3}\left(\frac{13\sqrt{13}}{8} - \frac{1}{8}\right) = \frac{\pi}{6}(13\sqrt{13} - 1).
\end{aligned}$$

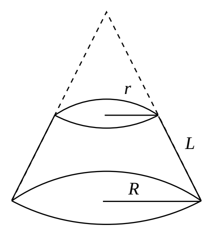
Exercises (for you to practice—not to turn in)

Exercise 1. Find the area of the surface obtained by revolving the graph of $y = \sqrt{2 - x}$ from $x = 0$ to $x = 2$ about the x-axis.

Exercise 2. Set up but do not evaluate an integral to compute the area of the surface obtained by revolving the graph of $y = e^x$ from $x = 0$ to $x = 1$ about the y-axis.

Derivation of (1). We need a formula from elementary plane geometry. A *frustum* is the solid obtained by removing from a given cone a smaller cone with the same apex. If F is a frustum with upper radius r, lower radius R, and slant height L, then the area of the lateral surface of F equals

$$\pi(r+R)L = 2\pi\left(\frac{r+R}{2}\right)L. \tag{2}$$



Now suppose that S is the surface obtained by revolving the graph of a nonnegative function $y = f(x)$ from $x = a$ to $x = b$ about the x-axis. Partition the interval $[a, b]$ into n equal subintervals

$$a = x_0 < x_1 < \cdots < x_n = b$$

of length $\Delta x = (b - a)/n$. As in the derivation of the arc length formula in section 7.1, the length of the line segment joining $(x_{i-1}, f(x_{i-1}))$ to $(x_i, f(x_i))$ is

$$L_i = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} = \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} \cdot \Delta x = \sqrt{1 + f'(c_i)^2} \cdot \Delta_x$$

for some c_i in the interval $[x_{i-1}, x_i]$. (This uses the Mean Value Theorem.)

The length L_i is the slant edge of the frustum F_i obtained by revolving this line segment about the x-axis. The radii r and R of the two bounding circles are $f(x_{i-1})$ and $f(x_i)$. By (2), the lateral surface area of F_i equals

$$2\pi\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right)\sqrt{1 + f'(c_i)^2} \cdot \Delta x$$

and so

$$\text{SurfArea}(S) = \lim_{n\to\infty}\sum_{i=1}^{n}\text{SurfArea}(F_i) = \lim_{n\to\infty} 2\pi\sum_{i=1}^{n}\left(\frac{f(x_{i-1}) + f(x_i)}{2}\right)\sqrt{1 + f'(c_i)^2} \cdot \Delta x. \tag{3}$$

This is not exactly a Riemann sum, since the expression $\frac{1}{2}(f(x_{i-1}) + f(x_i))\sqrt{1 + f'(c_i)^2}$ is not just a function of the point x_i (it depends on x_i, x_{i-1}, and some intermediate point c_i). However, for reasonably nice functions it can be checked that the value of this limit does not change if we replace the terms $f(x_{i-1})$ and $f(x_i)$ in (3) with $f(c_i)$. Then

$$\text{SurfArea}(S) = \lim_{n\to\infty} 2\pi\sum_{i=1}^{n} f(c_i)\sqrt{1 + f'(c_i)^2} \cdot \Delta x = 2\pi\int_a^b f(x)\sqrt{1 + f'(x)^2}\,dx$$

from the definition of the limit in terms of Riemann sums. □